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The handling of continuous barriers for derivatives on many underlyings

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Abstract

More and more structured (equity and FX) derivatives involve many underlyings subject to continuous barrier conditions. Known boundary value problems and high-dimensionalities are each by themselves amenable to specific numerical techniques that are virtually impossible to combine: finite-differencing for one and Monte Carlo for the other. In this presentation, I review some of the methods available to handle the situation when both features, continuous monitoring of barriers and high-dimensionality, are of crucial importance.

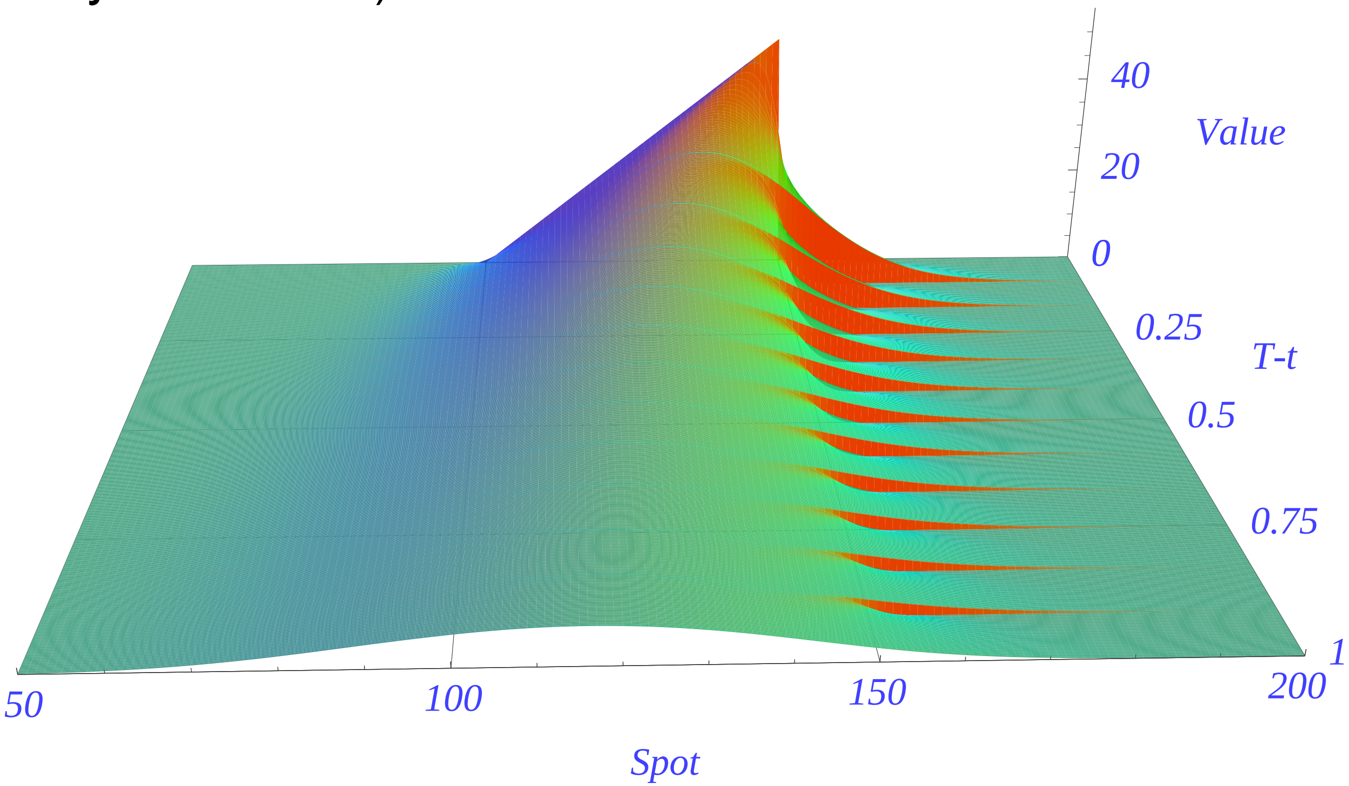
Overview and Introduction

- Review of continuous versus discrete monitoring
- Finite differencing methods
- Copula based approximations
- The Broadie-Glassermann-Kou [BGK99] approximation
- A conundrum

Review of continous versus discrete monitoring

There are many products involving barrier conditions:-

- Up-Out-, Down-Out, Up-In, Down-In- Calls and Puts (both disretely and continuously monitored)



- Average options: (discretely monitored)

Arithmetic	$\frac{\sum_i w_i S_i}{\sum_i w_i}$	Harmonic	$\frac{\sum_i w_i}{\sum_i \frac{w_i}{S_i}}$
Geometric	$\left(\prod_i S_i^{w_i}\right)^{\frac{1}{\sum_i w_i}}$	L_n norm	$\left(\sum_i S_i^n\right)^{\frac{1}{n}}$

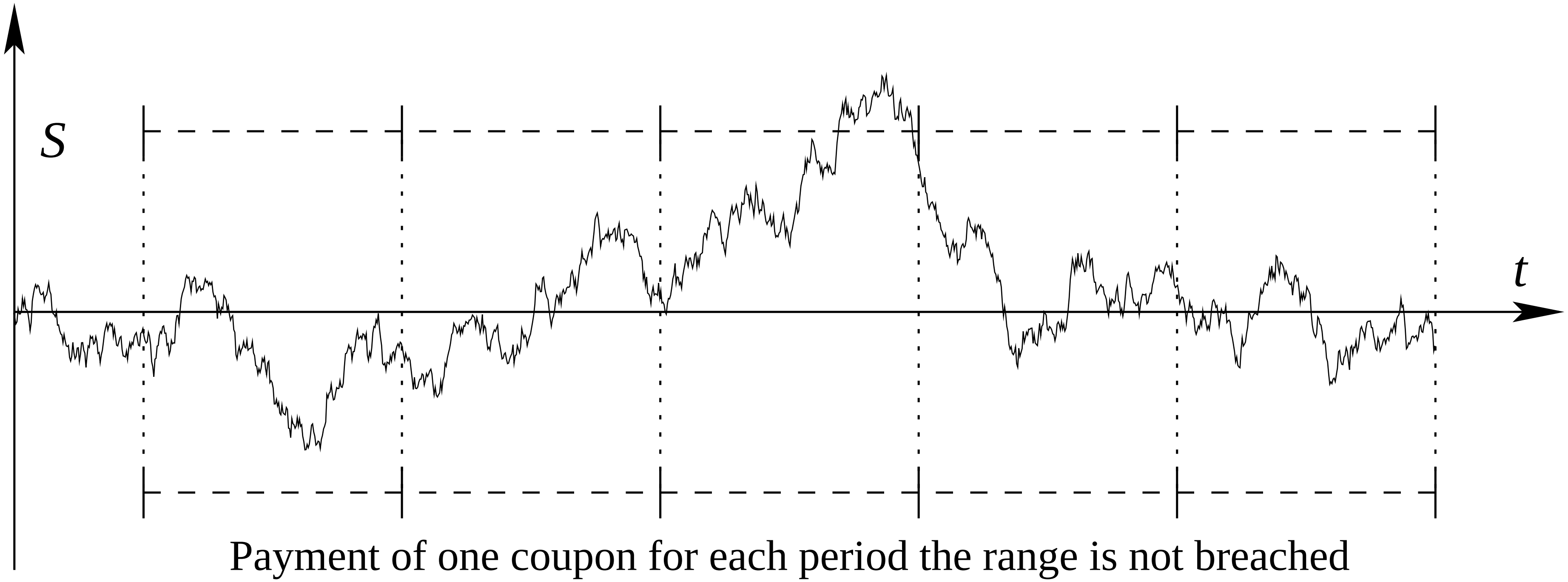
In themselves, they usually comprise no barrier features. However,

– $\max(S) = L_\infty$ and $\min(S) = L_{-\infty}$.

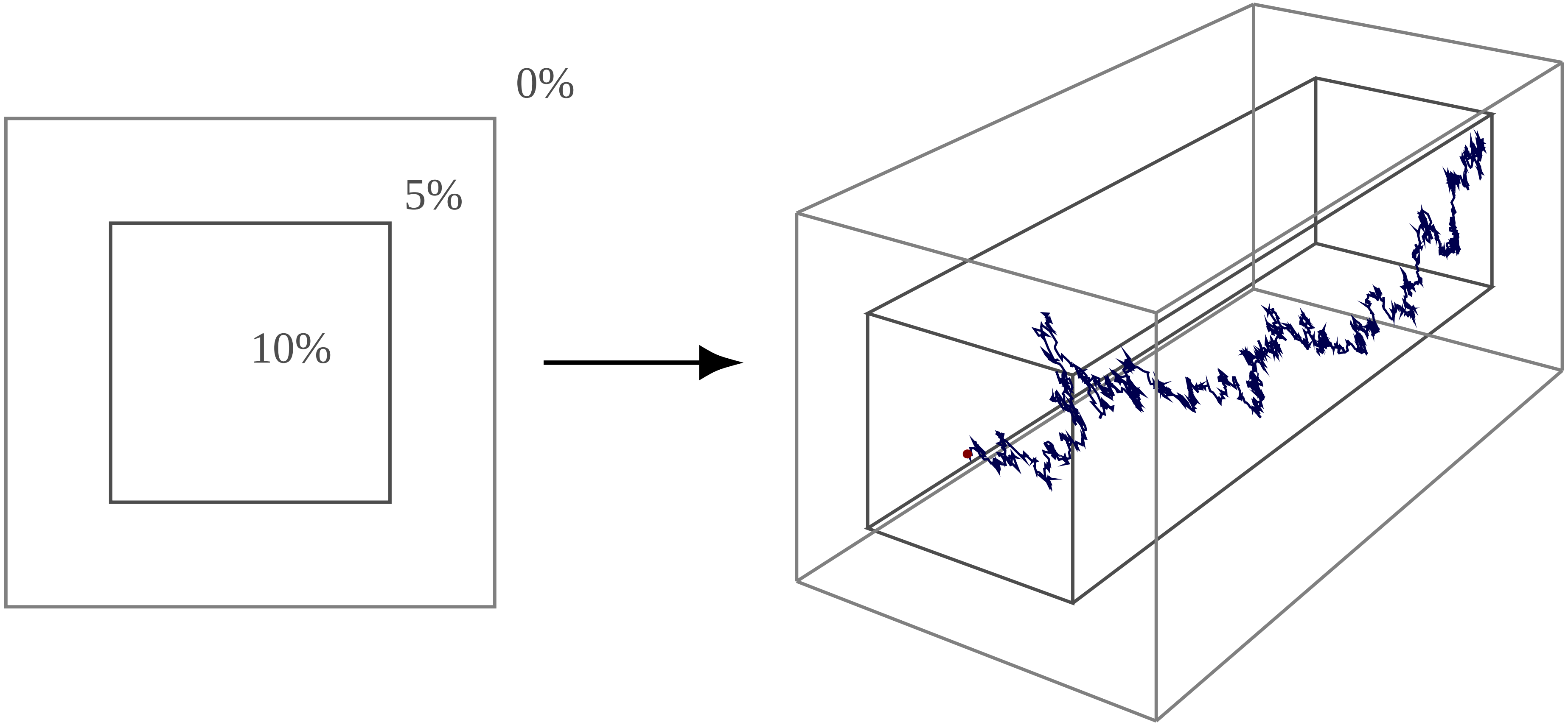
– they provide the building blocks for mountain range options with barrier features (Everest, Altiplano).

- Corridor style products:

  Range options: Payoff depends on time spent inside a range



Pyramid options: Payoff depends on how many of a set of nested corridors have been breached

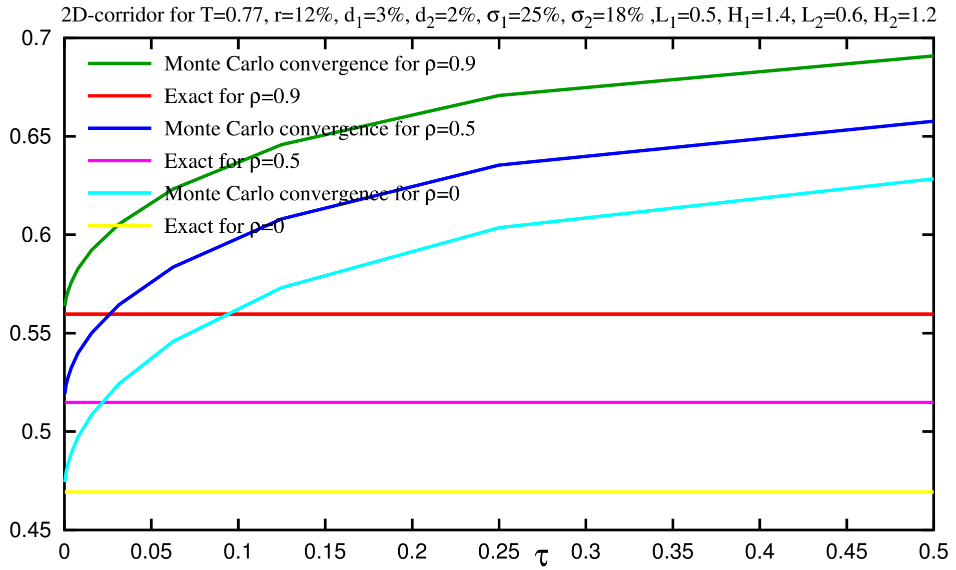


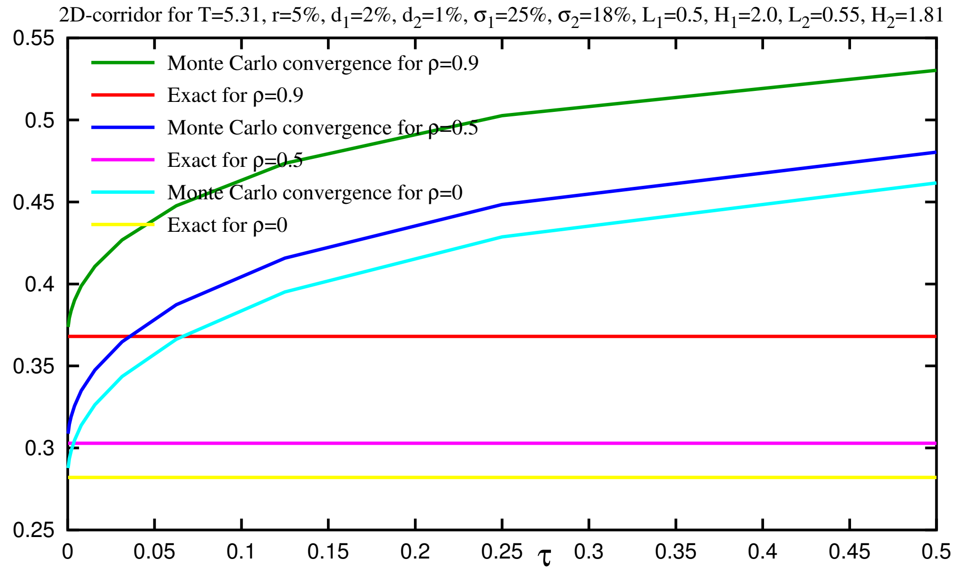
The connection between continuously and discretely monitored barrier product specifications appears to be rather innocuous: if you can price it as a discretely monitored proxy, just choose a fine time discretisation.

However, the convergence of a discretely monitored product V_{discrete} with monitoring period τ to the value of its continuous counterpart has a leading coefficient of order $\sqrt{\tau}$:

$$V_{\text{discrete}} = V_{\text{continuous}} + \mathcal{O}(\sqrt{\tau})$$

This can lead to a disproportionately large pricing error.





For single assets in the Black-Scholes framework, we can easily correct a discretely sampled Monte Carlo simulation for the conditional probability of a barrier breach occurring in-between monitoring times, *if we may assume constant drift coefficients (such as interest, dividend rates and volatilities) over the time step*:

The probability of not having exceeded a barrier at H conditional on the path being at S_t at time t and at $S_{t+\tau}$ at time $t+\tau$ is

$$p_{\text{conditional correction}} = 1 - \frac{\varphi(2h-z)}{\varphi(z)} \qquad \text{with} \qquad \varphi(z) = \frac{\mathrm{e}^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \tag{1}$$

and

h	z	
$\frac{\ln(H/S_t)}{\sigma\sqrt{\tau}}$	$\frac{\ln(S_{t+\tau}/S_t)}{\sigma\sqrt{\tau}}$	in the Black-Scholes (i.e. lognormal) model
$\frac{H-S_t}{\sigma\sqrt{\tau}}$	$\frac{S_{t+\tau}-S_t}{\sigma\sqrt{\tau}}$	in the Bachelier (i.e. normal) model

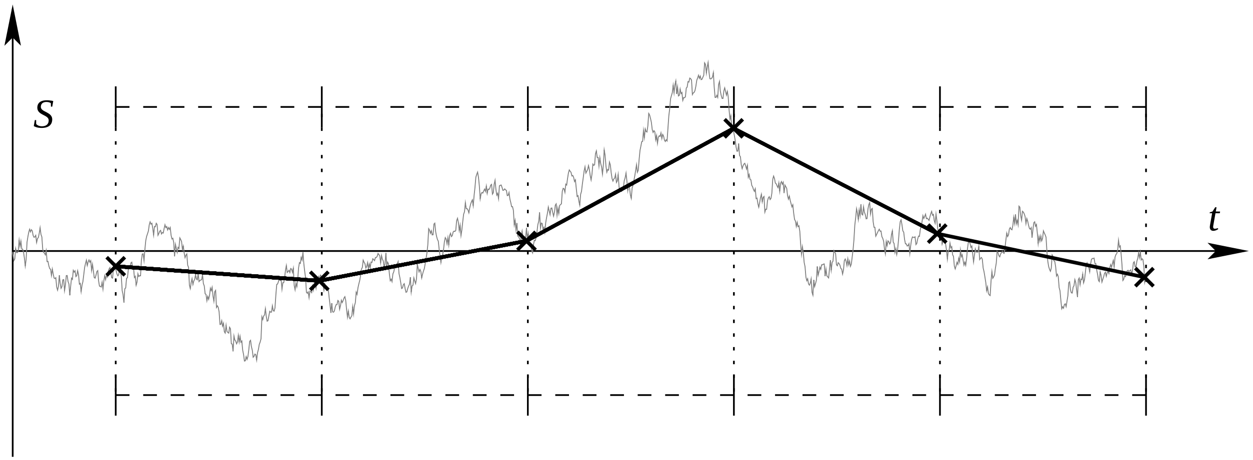
(2)

For double barrier (single asset) knock-out conditions, we can also correct *exactly* for the possibility of a barrier breach in between monitoring points by multiplying with the appropriate corrective probability

$$p_{\text{conditional correction}} = \psi_{\text{drift-free transition}}(S_t, S_{t+\tau}) / \varphi(z) \tag{3}$$

with $\psi_{\text{drift-free transition}}$ given in the appendix.

This is effectively a procedure of weighting the payoff for any one given path (that doesn't appear to knock-out on the discrete monitoring points) such that any path-dependent derivative is priced *exactly* as if continuous barriers had been imposed.



Finite differencing methods

The only problem with probability corrections for Monte Carlo methods is that they cannot be extended to multiple dimensions in the presence of correlation.

Finite-differencing methods allow for correlation dependence.

The least involved finite-differencing methods are ***explicit*** methods. They are comparatively easy to implement, even when many dimensions are needed.

The simplest explicit methods are so-called ***trees***. These allow little flexibility for the placement of nodes (in multiple dimensions, that is). The consequence is that, whilst trees work very well for problems with comparatively smooth terminal value conditions and gentle boundary constraints (including Bermudan or American style features), for barrier problems, they suffer from the node-jumping phenomenon which can spoil the result of a calculation significantly. This problem is so severe that, already in one dimension, there are dozens of publications on how to handle it. Little literature is available on how to handle this problem in more than one dimension.

The easiest finite-differencing method with some flexibility over the placement of nodes is the straight-forward explicit method. Like trees, however, its convergence order is $\mathcal{O}(\Delta t)$ and thus needs many steps for convergence reasons (if not for stability).

There are many explicit methods that give higher order convergence in Δt. Examples for this are the ***Lax-Wendroff*** method (identical to the ***predictor-corrector*** method as long as the drift terms don't cause characteristic lines to cross in between nodes connected by the scheme), and three time step schemes such as the ***leapfrog*** algorithm. These techniques give $\mathcal{O}(\Delta t^2)$ convergence.

The dominant problem with explicit methods is *stability*. Explicit schemes such as the plain vanilla method and the predictor-corrector approach require

$$\Delta t < \frac{1}{\sum_i \frac{\sigma_i^2}{\Delta x_i^2}} \tag{4}$$

(independent on the correlation). For a multi-dimensional corridor (read: *pyramid*) with comparatively narrow boundaries and a minimum of 100 nodes in each direction, this can mean that the scheme requires hundreds of thousands of time steps. However, since these methods are of convergence order $\mathcal{O}(\Delta x^2)$, it is usually possible to use very few spatial nodes indeed (it is possible to obtain reasonable accuracy with as little as 25 nodes in each direction).

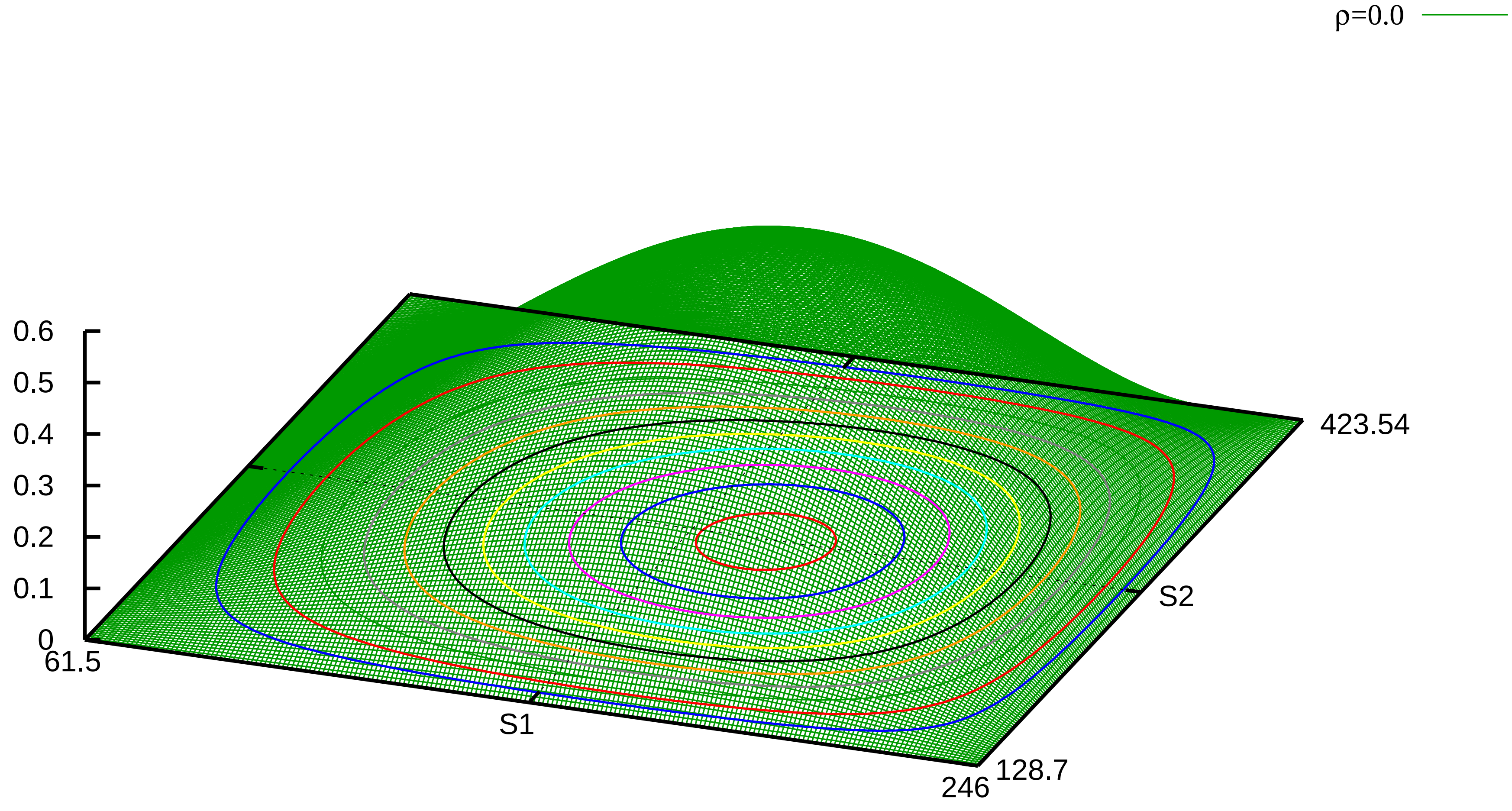
An exception to the stability problem of explicit methods is the leapfrog scheme. This three time step method is *unconditionally stable* given certain restrictions on the drift terms. It is of convergence order $\mathcal{O}(\Delta t^2)$, provided that both time steps are equal. Like all three time-step methods, it requires a startup at the beginning and whenever an arbitrary event time has to be accomodated. The leapfrog scheme can be extended to allow for different sized time steps whilst retaining $\mathcal{O}(\Delta t^2)$. However, in order not to lose stability, the second time step must be smaller than the first one which is yet another restriction we really don't need. What's more, this scheme allows for a *spurious* solution to arise. This phenomenon is conceptually somewhat similar to the feature of *ghosting* or *aliasing* sometimes seen in Fourier convolution methods with inadequate boundary handling, and is a side effect of the leapfrog scheme being in fact *marginally unconditionally stable*, i.e. it always has eigenvalues of unit absolute value.

Implicit methods connect several nodes on a known time slice with several nodes on a yet unknown time slice. The most popular, unconditionally stable, and $\mathcal{O}(\Delta t^2)$ convergence order method is the Crank-Nicolson algorithm, which retains all of the above properties in multiple dimensions. In more than one dimension, however, this requires the numerical solution of a large matrix problem at each time step which is (arguably) best handled using quasi minimal residual (QMR) or biconjugate gradient (BCG) methods with preconditioning and stabilisation, or multigrid methods. *QMR, BCG, and multigrid methods are the most commonly used ones in meteorology and other sciences and engineering disciplines employing numerical techniques for the solution of convection-diffusion problems.*

There are many hybrid methods, Alternating Direction Implicit being but one of them. The idea is to use the Crank-Nicolson method only in one direction, but switch which direction has it applied. Unfortunately, this method also has its problems. The main one is that it requires a coordinate transformation that removes the correlation terms of the governing partial differential equation. Even when this is possible, it spoils the nice alignment of nodes with barrier conditions that is possible without the transformation.

To summarise: if you need to accurately price products with continuous barrier features on a relatively small number of assets for a model that allows the formulation of a partial differential equation, the best approach is (arguably) to invest in the development of finite-differencing solvers using QMR, BCG, or multigrid methods.

Survival density of 2D-corridor for T=0.77, r=12%, d_1=3%, d_2=2%, σ_1=25%, σ_2=18%, L_1/S_1=0.5, H_1/S_1=1.4, L_2/S_2=0.6, H_2/S_2=1.2



Arrival density conditional on no barrier breach as a function of logarithm of terminal spot values

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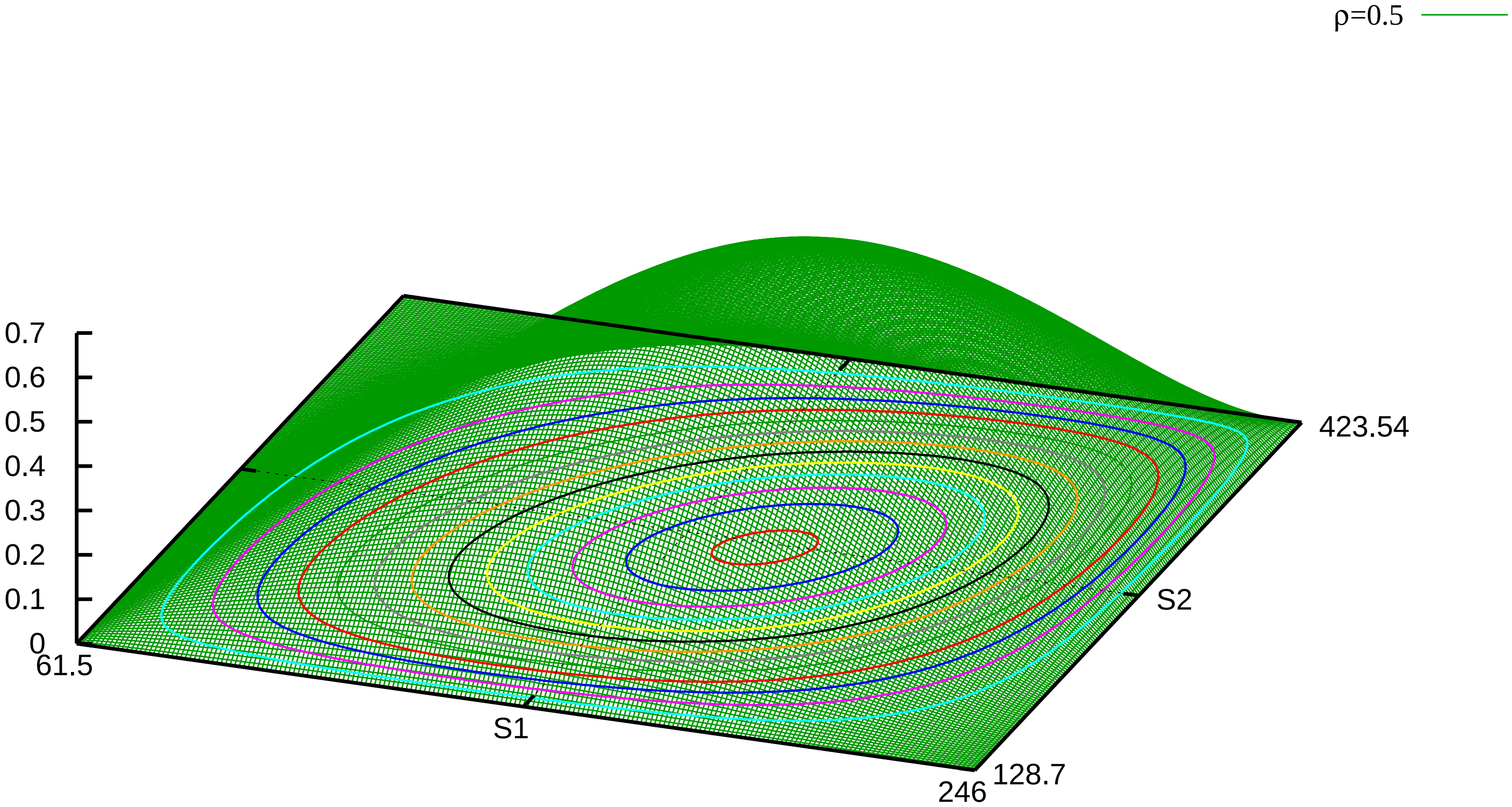

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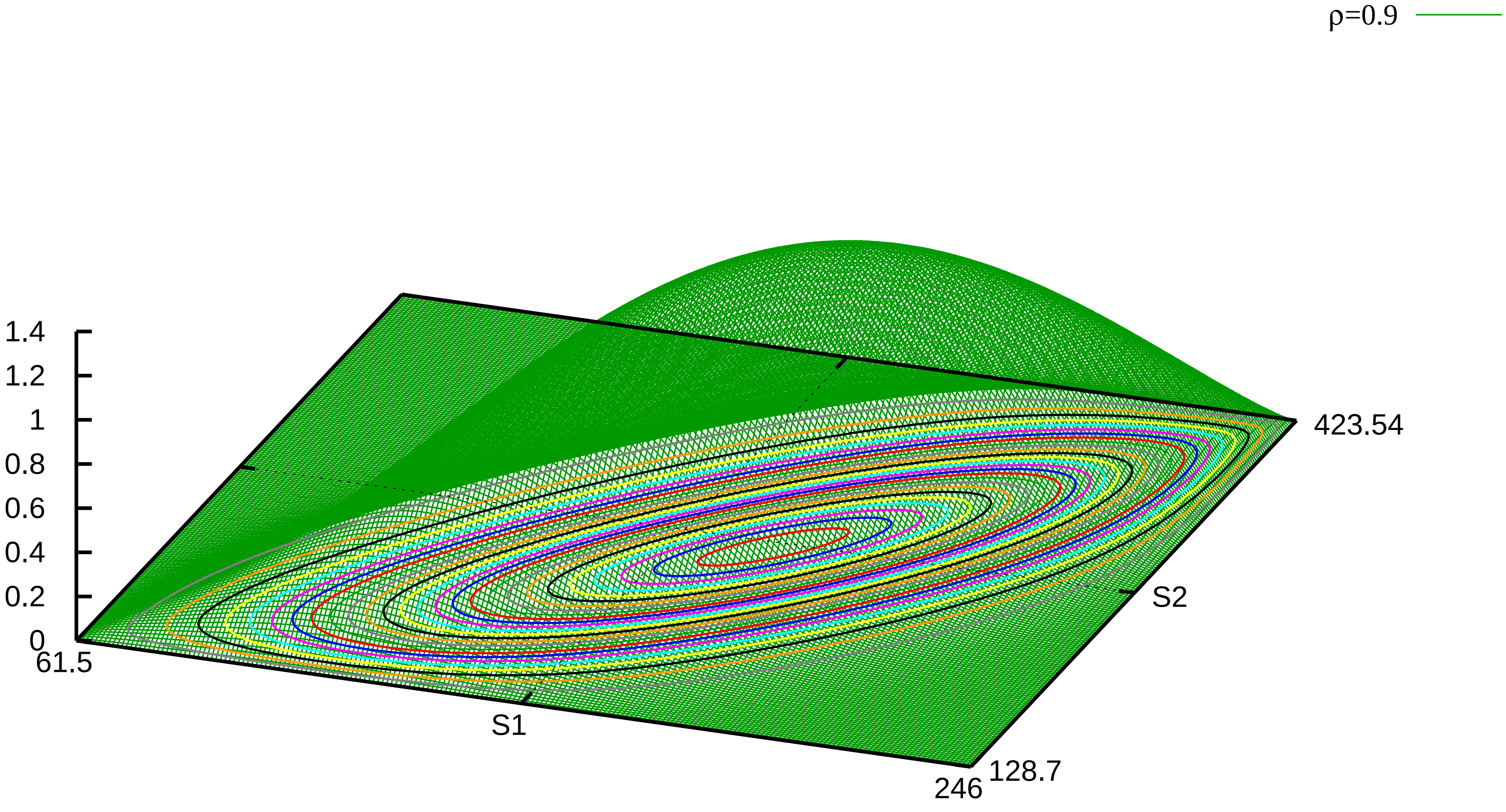
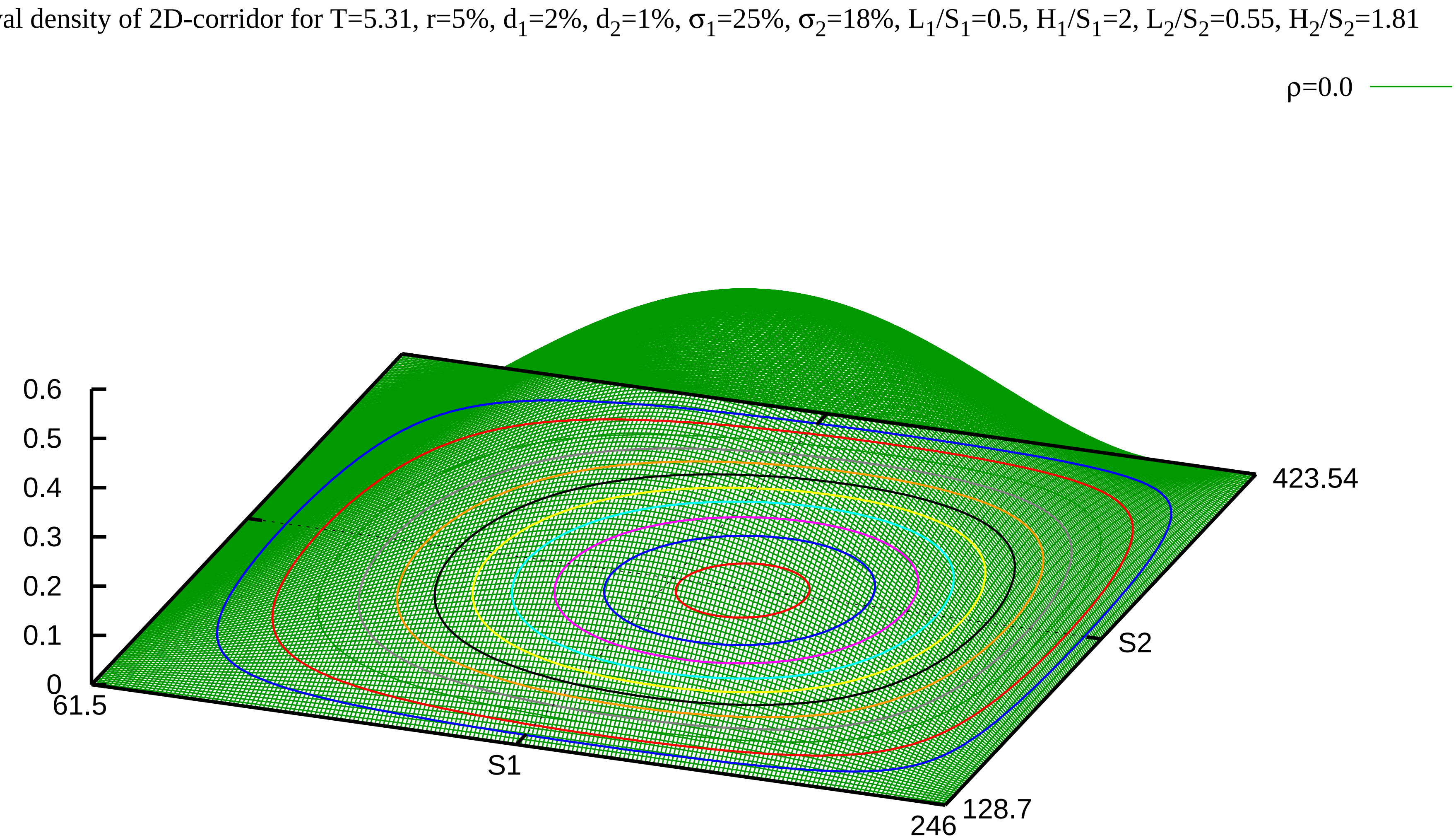
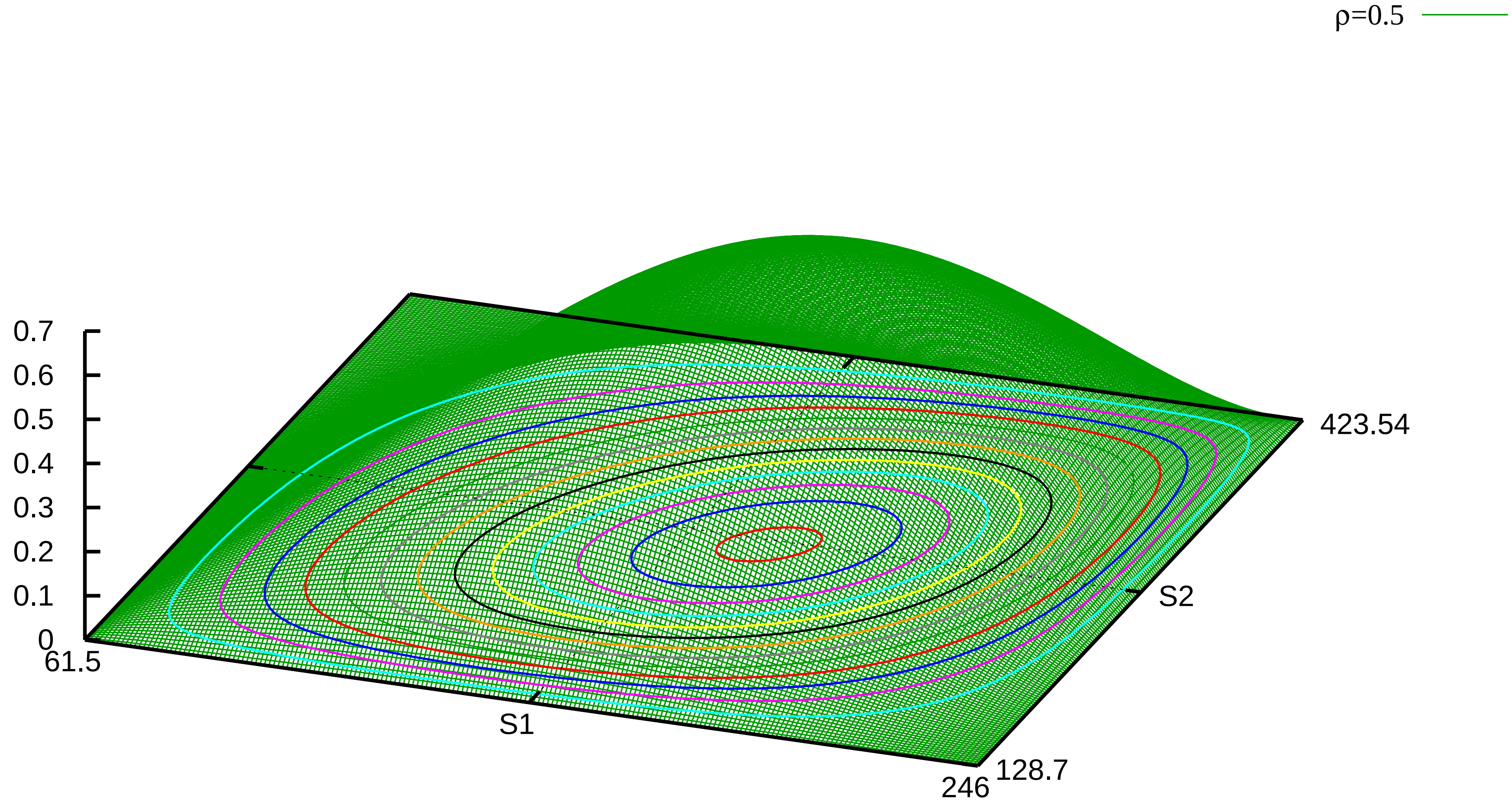


Arrival density conditional on no barrier breach as a function of logarithm of terminal spot values



Arrival density conditional on no barrier breach as a function of logarithm of terminal spot values

Survival density of 2D-corridor for T=5.31, r=5%, d_1=2%, d_2=1%, σ_1=25%, σ_2=18%, L_1/S_1=0.5, H_1/S_1=2, L_2/S_2=0.55, H_2/S_2=1.81



Arrival density conditional on no barrier breach as a function of logarithm of terminal spot values

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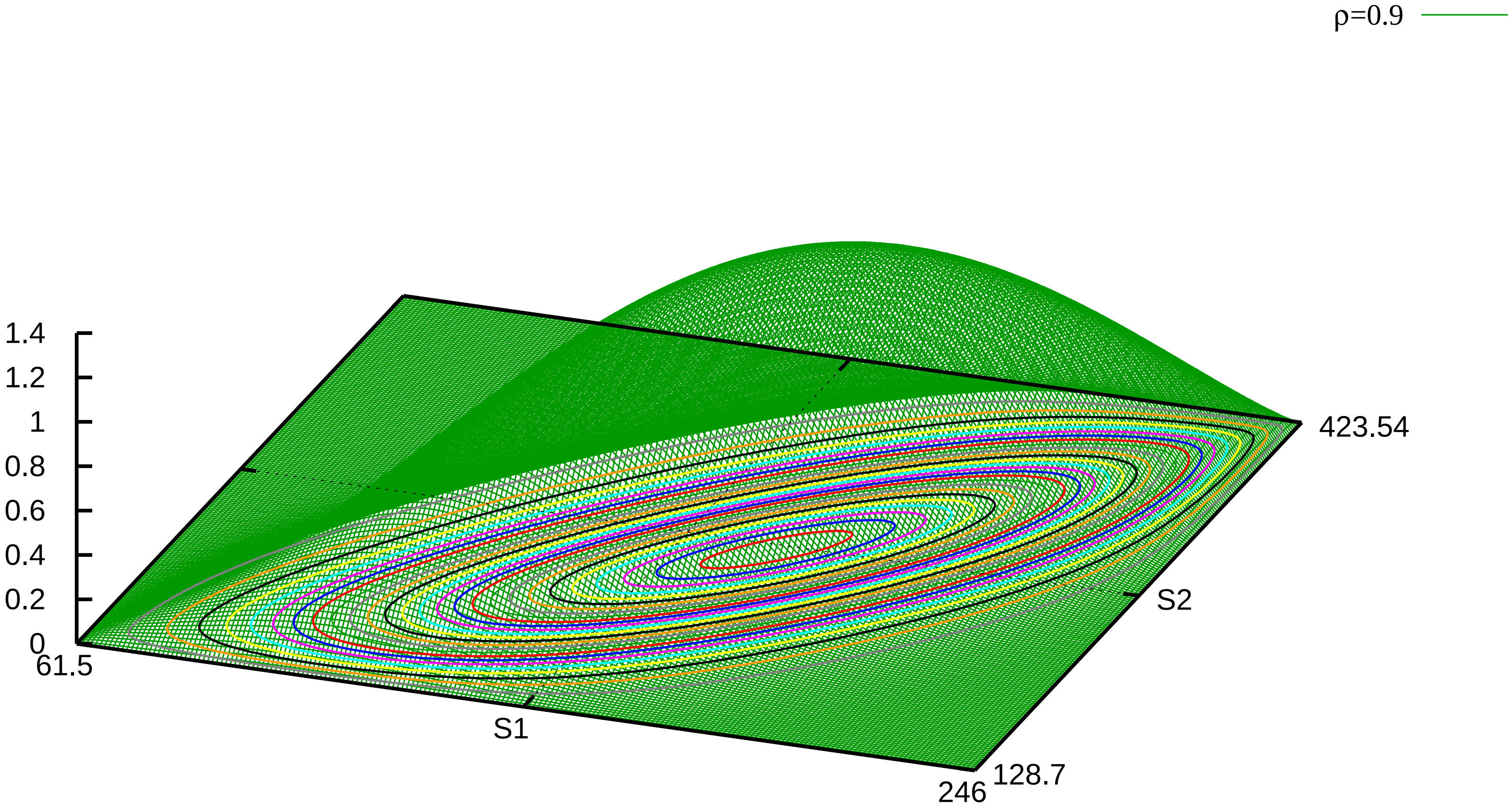


Arrival density conditional on no barrier breach as a function of logarithm of terminal spot values

Copula based approximations

A copula is a generic method to connect two otherwise independent *marginal* densities to form a joint multivariate density.

Can we use this technique to approximate multivariate survival densities as they result from hard problems such as the multidimensional corridor?

Let's take the example of the 2D corridor marginals connected by a Gaussian copula.

Define

$$x_i = \ln S_i(T), \qquad \xi_i = \frac{\xi_i - \ln S_i(0)}{\sigma_i \sqrt{T}}, \qquad \Psi_i(\xi_i) = \int_{-\infty}^{\xi_i} \psi_i(\zeta) \mathrm{d}\zeta \qquad \text{for} \quad i = 1, 2$$

with $\psi_i(\xi_i)$ being the marginal (drift-adjusted) density as a function of the regularised logarithm ξ_i of terminal spot value S_i.

Also, denote the univariate cumulative normal distribution as $\Phi(\cdot)$ and the bivariate cumulative normal distribution of x and y with correlation ρ as $\Phi(x, y, \rho)$.

The Gaussian copula is a $[0, 1]^2 \to [0, 1]$ function given by

$$C_{\text{Gaussian}}(u_1, u_2, \rho) = \Phi(z_1, z_2, \rho) \qquad \text{with} \qquad z_i = \Phi^{-1}(u_i) \ . \tag{5}$$

The uniform variates u_1 and u_2 are linked to the marginal densities by

$$u_i = \Psi_i(\xi_i) \ . \tag{6}$$

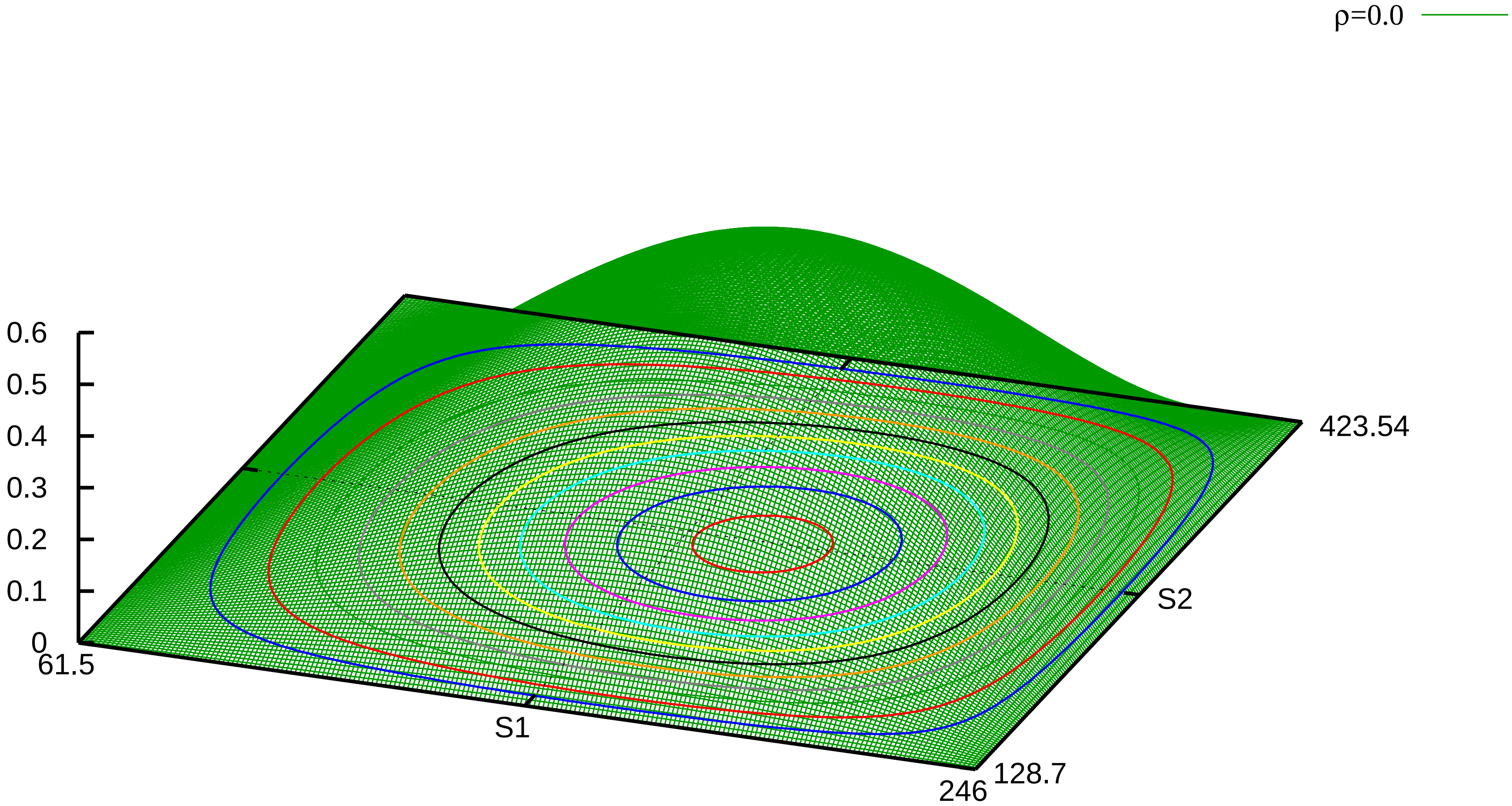
This enables us to compute the joint density as generated by the marginals linked with the Gaussian copula:

$$\psi(x_1, x_2) = \frac{\varphi(z_1, z_2, \rho)}{\varphi(z_1)\varphi(z_2)} \cdot \frac{\psi_1(\xi_1)\psi_2(\xi_2)}{\sigma_1\sqrt{T}\sigma_2\sqrt{T}} \tag{7}$$

with

$$\varphi(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \, \mathrm{e}^{-\frac{1}{2}\left(\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right)} \ .$$

Copula generated density of 2D-corridor for T=5.31, r=5%, d_1=2%, d_2=1%, σ_1=25%, σ_2=18%, L_1/S_1=0.5, H_1/S_1=2, L_2/S_2=0.55, H_2/S_2=1.81



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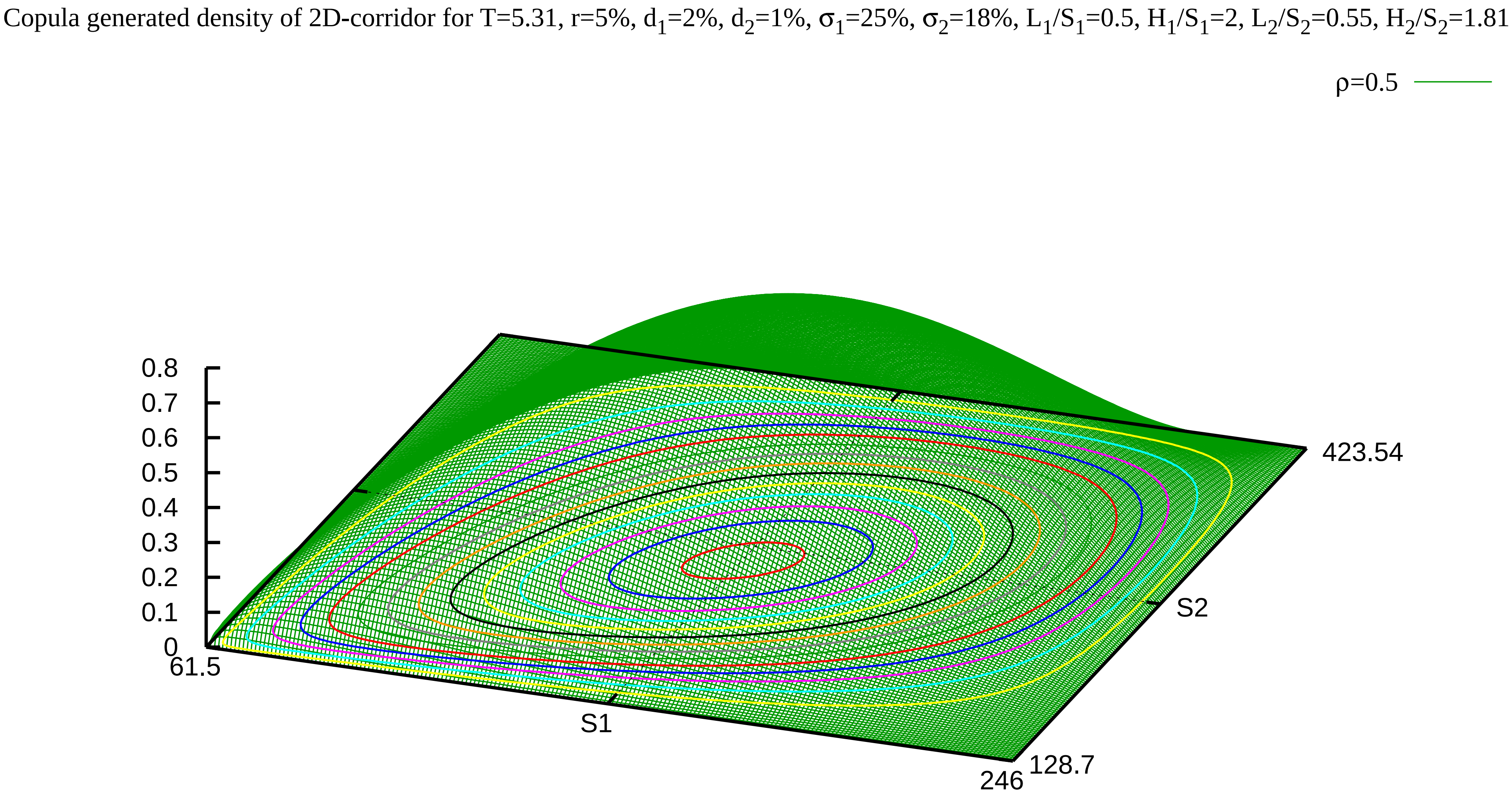
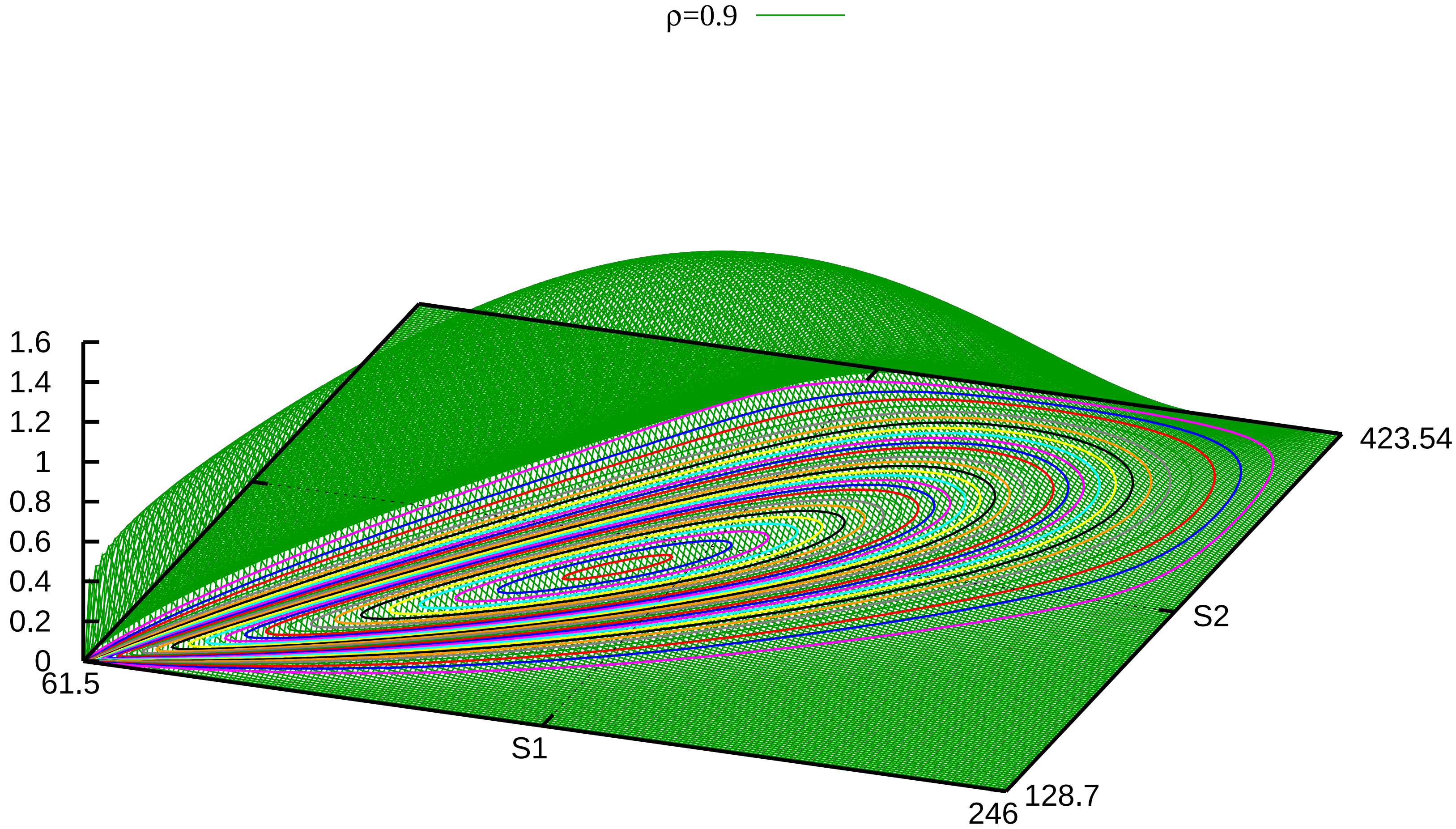

Copula generated density of 2D-corridor for T=5.31, r=5%, d1=2%, d2=1%, σ1=25%, σ2=18%, L1/S1=0.5, H1/S1=2, L2/S2=0.55, H2/S2=1.81
ρ=0.5
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
61.5
S1
246
128.7
S2
423.54

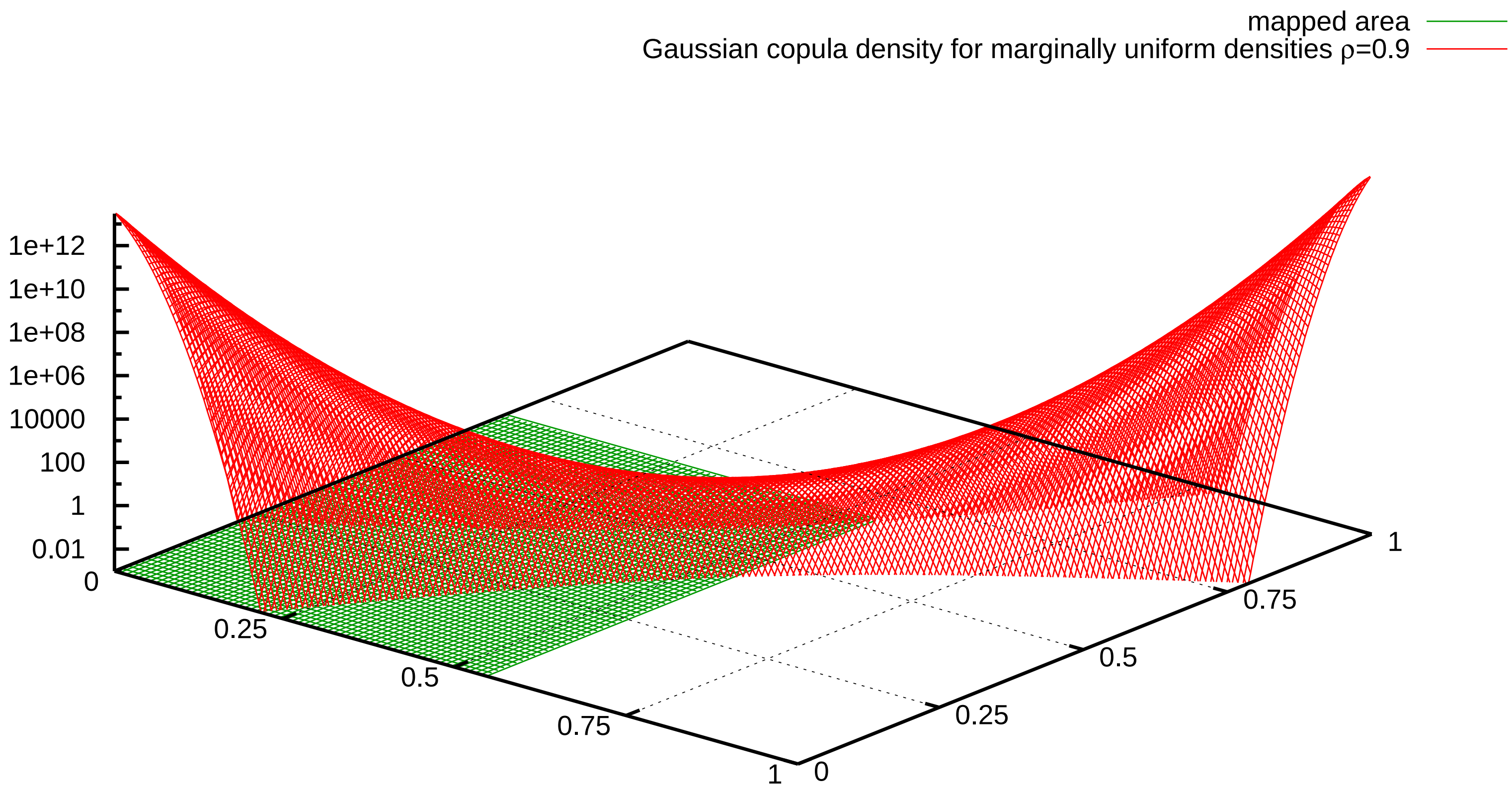
Arr

Copula generated density of 2D-corridor for T=5.31, r=5%, d_1=2%, d_2=1%, σ_1=25%, σ_2=18%, L_1/S_1=0.5, H_1/S_1=2, L_2/S_2=0.55, H_2/S_2=1.81



Arr

The pronounced cornering effect we see is due to the fact that the given marginal densities cover only part of the uniform interval [0,1] which is then connected via the Gaussian copula:



Note the logarithmic scale. The base grid actually only covers the interval $[\Phi(-8), \Phi(8)]^2$, i.e. not exactly $[0, 1]^2$.

It is possible to improve the copula generated density approximation by using a shifted and rescaled area of the uniform Gaussian copula density for the transformation. This means, we set

$$l_i = \frac{\ln\left(L_i / S_i(0)\right)}{\sigma_i\sqrt{T}}, \qquad h_i = \frac{\ln\left(H_i / S_i(0)\right)}{\sigma_i\sqrt{T}}, \qquad p_i = \Psi_i(\infty) = \Psi_i(h_i),$$

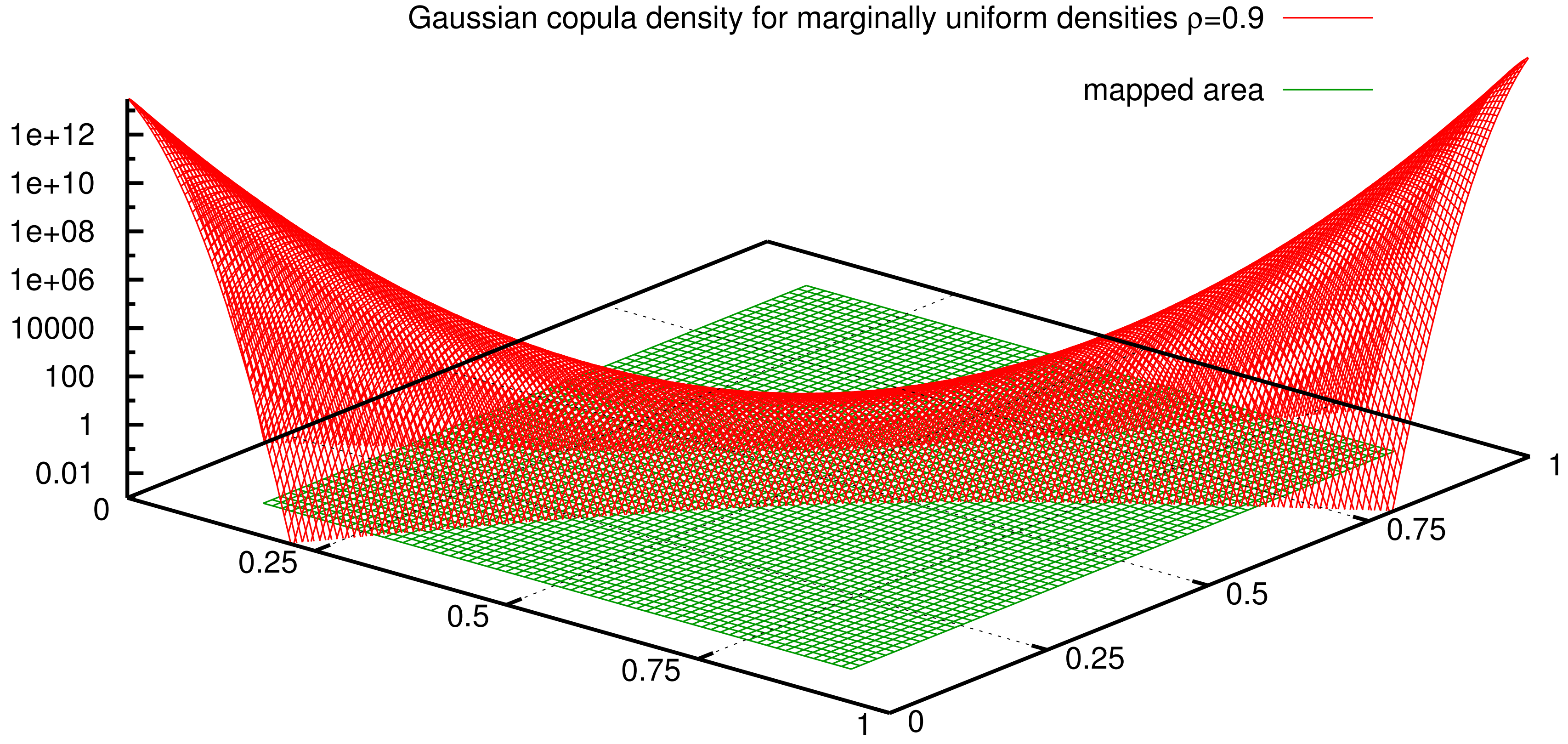
$$a_i = \Phi(l_i), \qquad b_i = \frac{\Phi(h_i) - \Phi(l_i)}{p_i}$$

and set

$$z_i = \Phi^{-1}(a_i + b_i u_i) \tag{8}$$

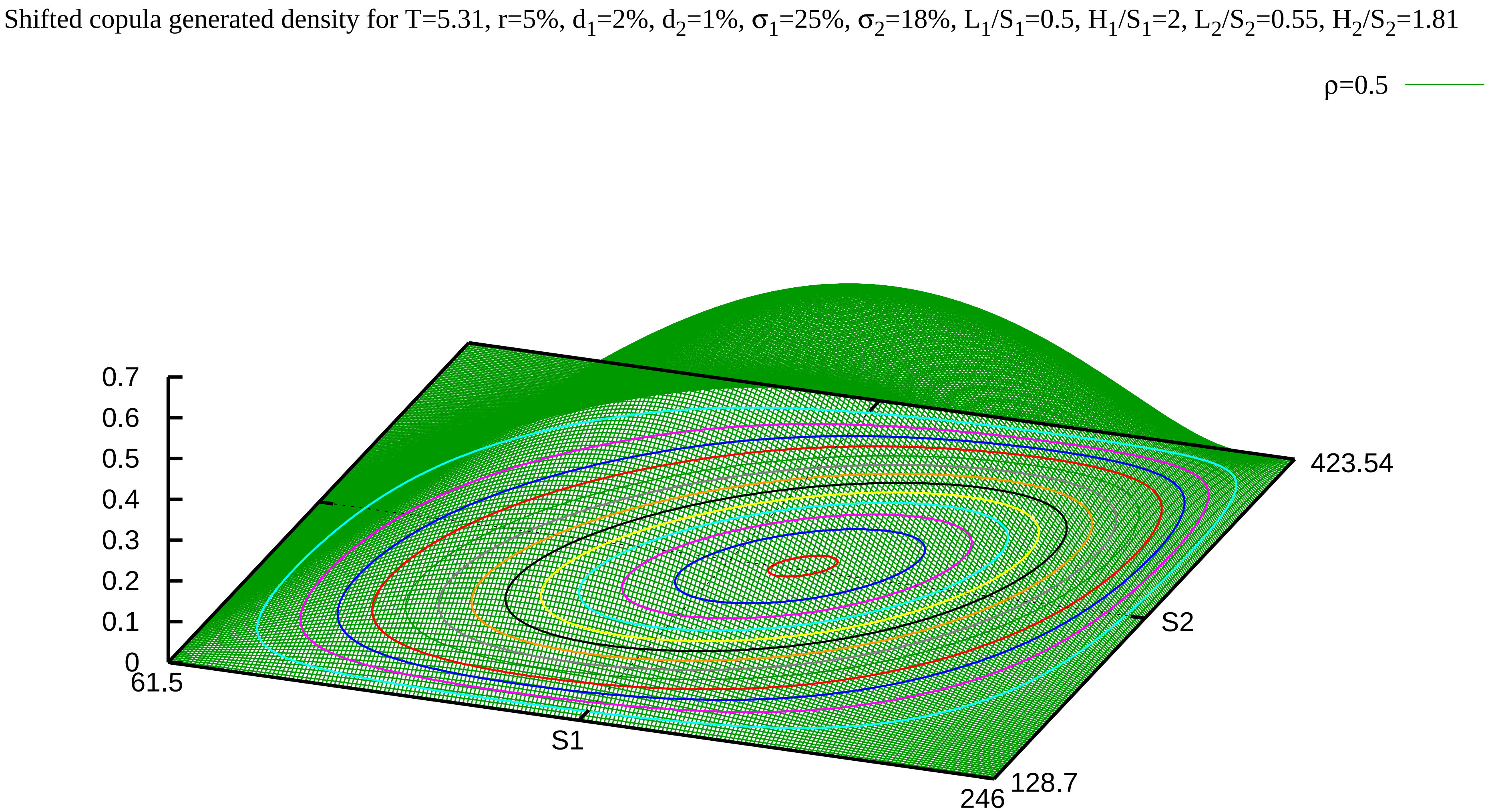
to be used in (7).

The shifted and rescaled mapping area of the Gaussian copula is more suitable for the generation of the joint density approximation:

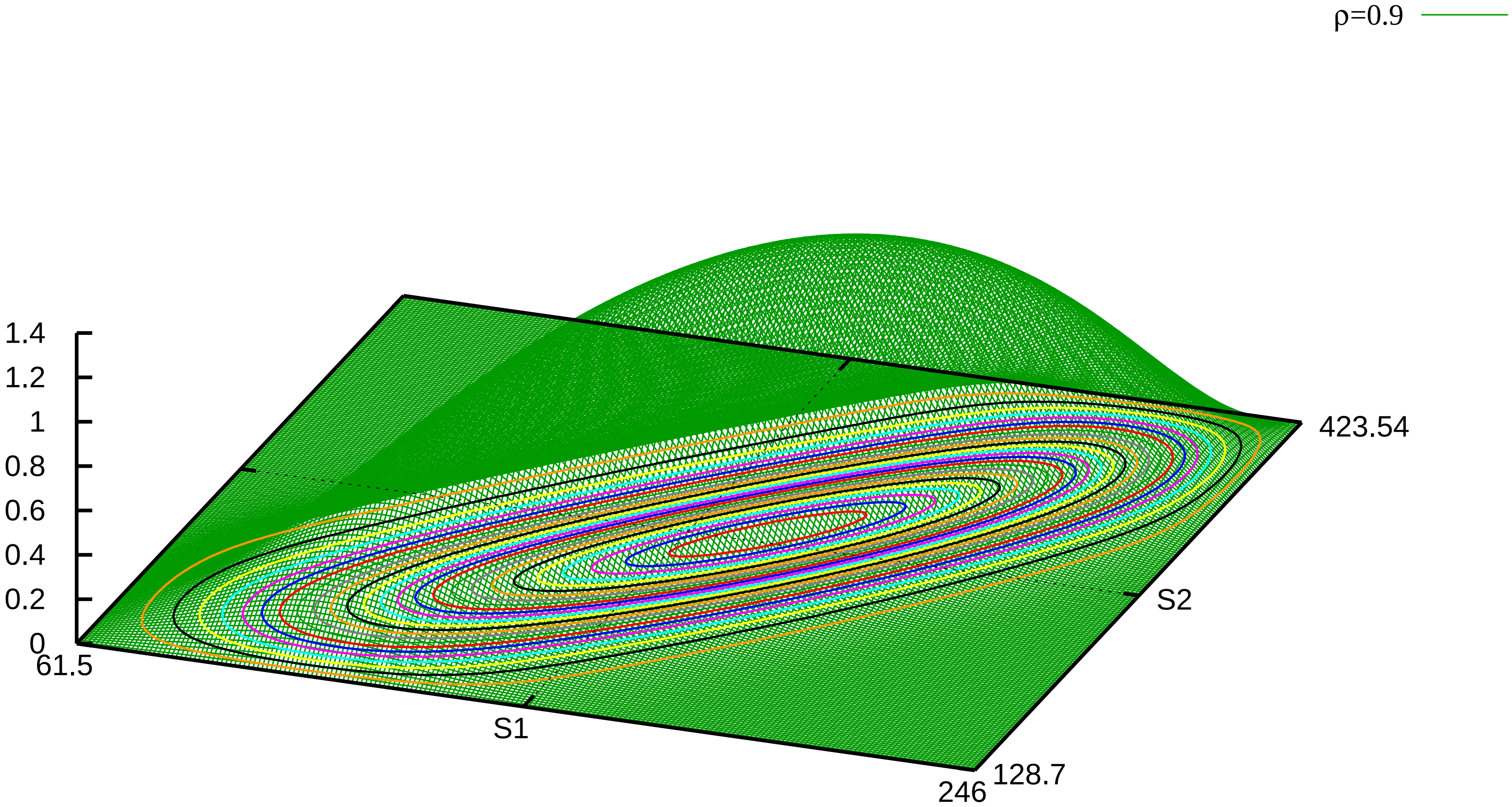


The shifted and rescaled mapping area avoids the singularities at the corners.

This results in better approximate copula generated densities.



Shifted copula generated density for T=5.31, r=5%, d_1=2%, d_2=1%, σ_1=25%, σ_2=18%, L_1/S_1=0.5, H_1/S_1=2, L_2/S_2=0.55, H_2/S_2=1.81



Possible applications of copula-generated joint densities subject to barrier conditions:

- High-dimensional products with knock-out boundaries where all other methods become prohibitively expensive.

- Medium-dimensional products with moderate dependence on accurate modelling of corner densitiies

The computational advantage of copula generated joint survival densities is that prices can be computed using a low-dimensional Monte Carlo simulation.

For a standard pyramid product, for instance, the Monte Carlo dimensionality reduces from 108 (3 years, monthly barrier points, 3 assets) to 6 (two corridors in 3 asset dimensions).

The Broadie-Glassermann-Kou approximation

In 1996, Broadie, Glassermann, and Kou [BGK99] developed an expansion that provides an approximate connection between the price of a regularly discretely monitored barrier product and the associated analytical pricing formula of the continuous version of the derivative contract.

Let the continuous pricing formula for a barrier product with continuous barrier level at B be given by

$$V_{\mathsf{c}} = F(B)$$

In the Black-Scholes framework, the BGK approximation for the price of a product whose barrier is discretely monitored but that is otherwise identical is then given by

$$V_{\mathsf{d}} \;\approx\; F(\mathrm{e}^{\pm\frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}}\sigma\sqrt{\tau}} \cdot B) + \mathcal{O}(\tau)\ . \tag{9}$$

with $\frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}} \;\approx=\; 0.58259716$. The only conditions on the expansion are that τ must be not too large and that the spot must be sufficiently far away from the barrier.

In other words, the discretely monitored contract is approximated as a continuously monitored contract with a shifted barrier level given by

$$B_{\text{shifted}} \;=\; \mathrm{e}^{\pm\frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}}\sigma\sqrt{\tau}} \cdot B\;. \tag{10}$$

The sign in the exponent in equations (9) and (10) is selected according to whether the initial spot level is above or below the threshold barrier.

In the normal (Bachelier) setting, the barrier shift is given by

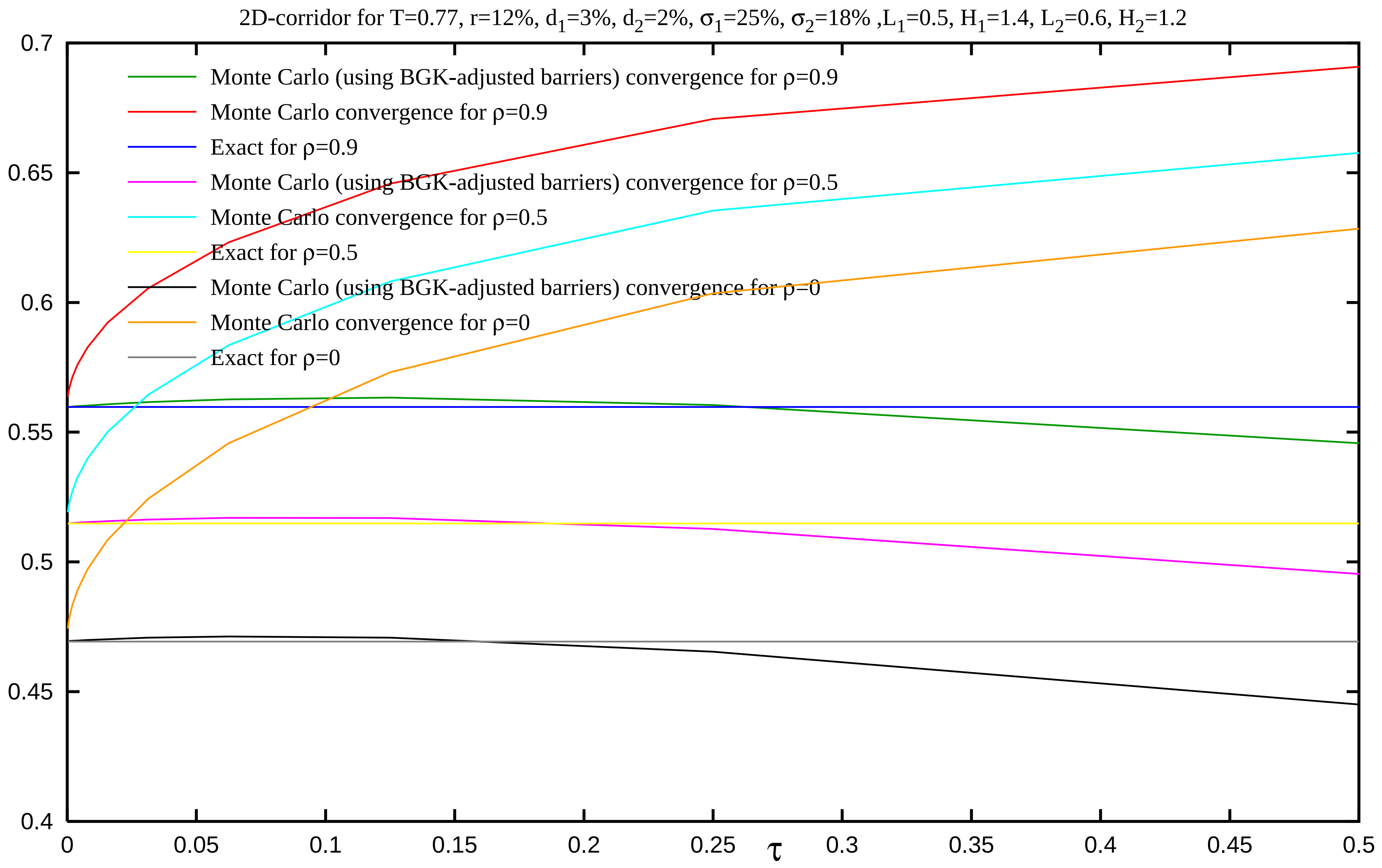
$$B_{\text{shifted}} \;=\; B \pm \frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}}\sigma\sqrt{\tau}\;. \tag{11}$$

The BGK formula can also be used in reverse to approximate an adjusted barrier to be used in a time-discretised Monte Carlo simulation with monitoring points being τ when we actually wish to compute the value of a continuously monitored barrier product:

$$B_{\text{shifted for Monte Carlo}} \;=\; \mathrm{e}^{\mp\frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}}\sigma\sqrt{\tau}} \cdot B \; . \tag{12}$$

Note: The BGK adjustment is an expansion in the monitoring interval τ and thus *not affected by correlation*.

The BGK barrier adjustment is an extremely important and powerful tool for the handling of continuous barrier features.


2D-corridor for T=0.77, r=12%, d1=3%, d2=2%, σ1=25%, σ2=18% ,L1=0.5, H1=1.4, L2=0.6, H2=1.2
Monte Carlo (using BGK-adjusted barriers) convergence for ρ=0.9
Monte Carlo convergence for ρ=0.9
Exact for ρ=0.9
Monte Carlo (using BGK-adjusted barriers) convergence for ρ=0.5
Monte Carlo convergence for ρ=0.5
Exact for ρ=0.5
Monte Carlo (using BGK-adjusted barriers) convergence for ρ=0
Monte Carlo convergence for ρ=0
Exact for ρ=0
0.7
0.65
0.6
0.55
0.5
0.45
0.4
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τ
0.3
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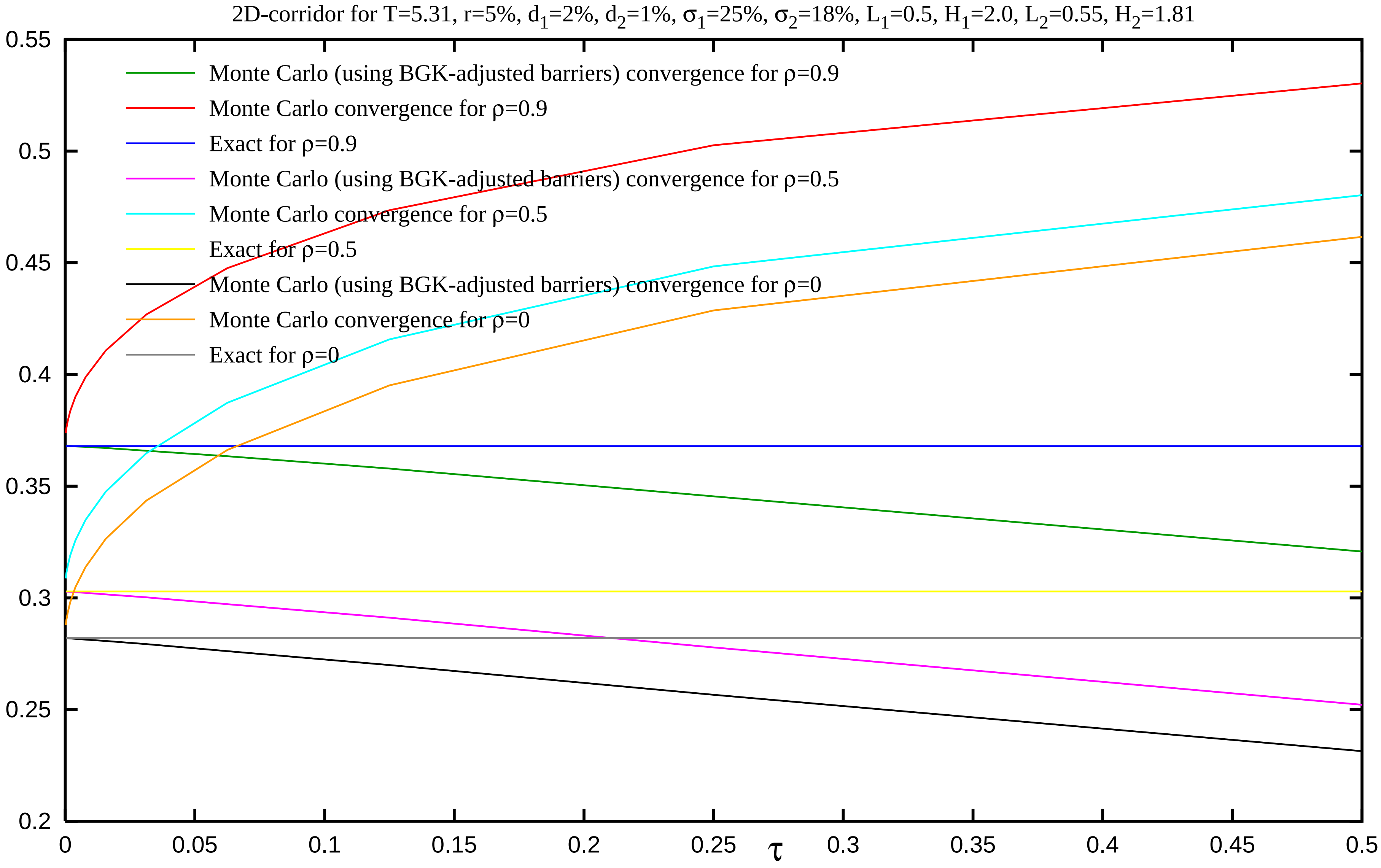
0.45
0.5


2D-corridor for T=5.31, r=5%, d1=2%, d2=1%, σ1=25%, σ2=18%, L1=0.5, H1=2.0, L2=0.55, H2=1.81
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Monte Carlo convergence for ρ=0
Exact for ρ=0
0.55
0.5
0.45
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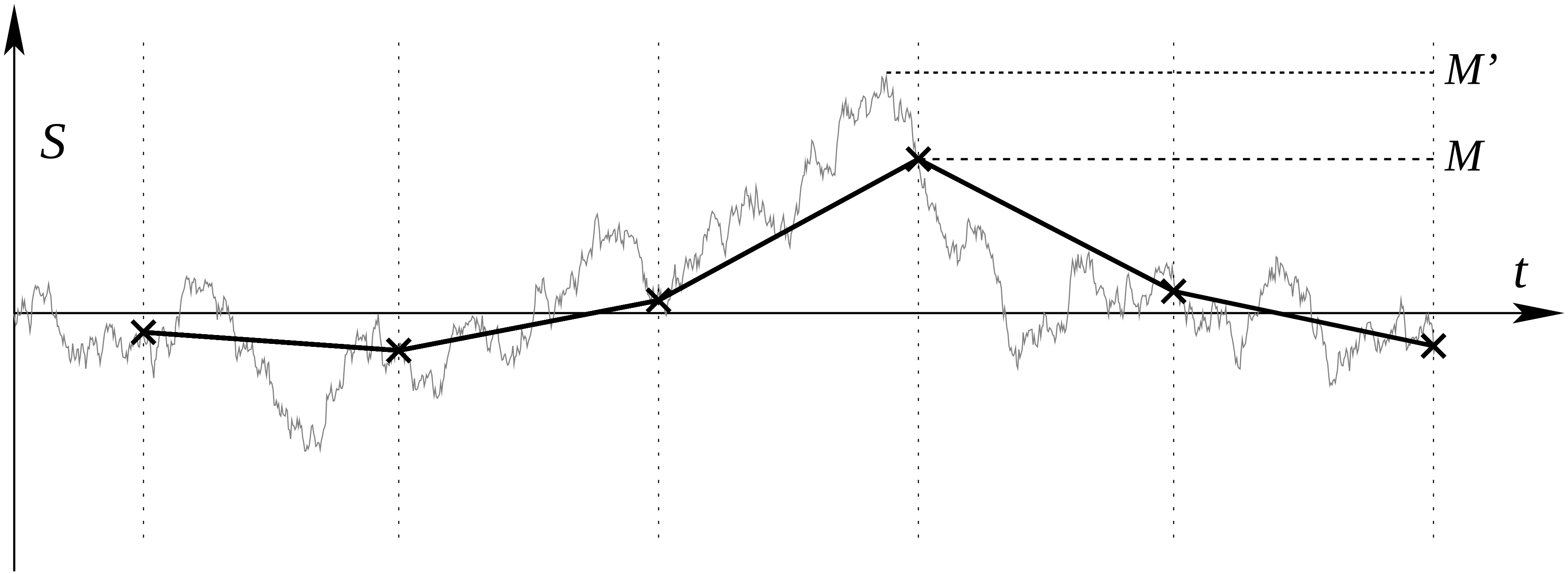
0.35
0.3
0.25
0.2
0
0.05
0.1
0.15
0.2
0.25
τ
0.3
0.35
0.4
0.45
0.5

An alternative view is to see the BGK formula as an adjustment to the path-maximum (-minimum) along any one Monte Carlo path in order to compensate for the fact that the maximum (minimum) of any continuous path is higher (lower) than the highest (lowest) point observed at any of the time-discretised monitoring points.

This enables us to use the same approximation for the Monte Carlo simulation of products that depend on the maximum or minimum over a given period. This is useful for the pricing of products of lookback and hindsight style.



A simple model for the skew is *displaced diffusion* [Rub83], where the spot process is governed by the stochastic differential equation

$$\frac{\mathrm{d}(S+A)}{S+A} = \mu \mathrm{d}t + \sigma_{\mathsf{dd}} \mathrm{d}W \tag{13}$$

whose solution is

$$S_t = (S_0 + A_0)\mathrm{e}^{(\mu - \frac{1}{2}\sigma_{\mathsf{dd}})t + \sigma_{\mathsf{dd}} W_t} - A_t \tag{14}$$

with $A_t = A_0 \mathrm{e}^{\mu t}$. An approximate barrier adjustment in this case is

$$B_t \; = \; (B + A_t)\mathrm{e}^{\mp \frac{|\zeta(\frac{1}{2})|}{\sqrt{2\pi}}\sigma\sqrt{\tau}} - A_t \; . \tag{15}$$

This means, we need to adjust the barrier level individually for each discrete monitoring time unless the risk-neutral drift μ is equal to zero.

A conundrum

The BGK shift relies on the initial spot level not to be too close to the barrier.

What happens if we get near the barrier?

For example, take a drift-free standard Wiener process W on the time interval $[0, 1]$.

The probability of survival given a continuous absorbing barrier at $h_{\mathrm{c}} \geq 0$ is

$$p_{\text{survival}} = 1 - 2\Phi(-h_{\mathrm{c}}) \ . \tag{16}$$

Now, assume we have $n \geq 1$ (i.e. $\tau = 1/n$) discrete monitoring points at the discrete monitoring level $h_{\mathrm{d}} \geq 0$. Absorption occurs if the process is below the barrier at any of the monitoring times.

If we can compute the survival probability $p_{\mathsf{survival}}(h_{\mathsf{d}};\tau)$ as a function of τ for a number of different monitoring discretisations τ, we can infer the associated equivalent continuous barrier by virtue of

$$h_c(h_{\mathsf{d}};\tau) = \Phi^{-1}\left(\tfrac{1}{2} - \tfrac{1}{2}p_{\mathsf{survival}}(h_{\mathsf{d}};\tau)\right) \ . \tag{17}$$

Let us also posit the existence of an equivalent continuous barrier correction expansion

$$h_c(h_{\mathsf{d}};\tau) \ = \ \sum_{n=0}^{\infty} f_i(h_{\mathsf{d}}) \cdot \tau^{\,n/2} \ . \tag{18}$$

Naturally, we have $f_0(x) \equiv x$, i.e.

$$h_c(h_{\mathsf{d}};\tau) \ = \ h_{\mathsf{d}} + \sum_{n=1}^{\infty} f_i(h_{\mathsf{d}}) \cdot \tau^{\,n/2} \ .$$

In general, we can compute $f_1(h_{\mathsf{d}})$ using

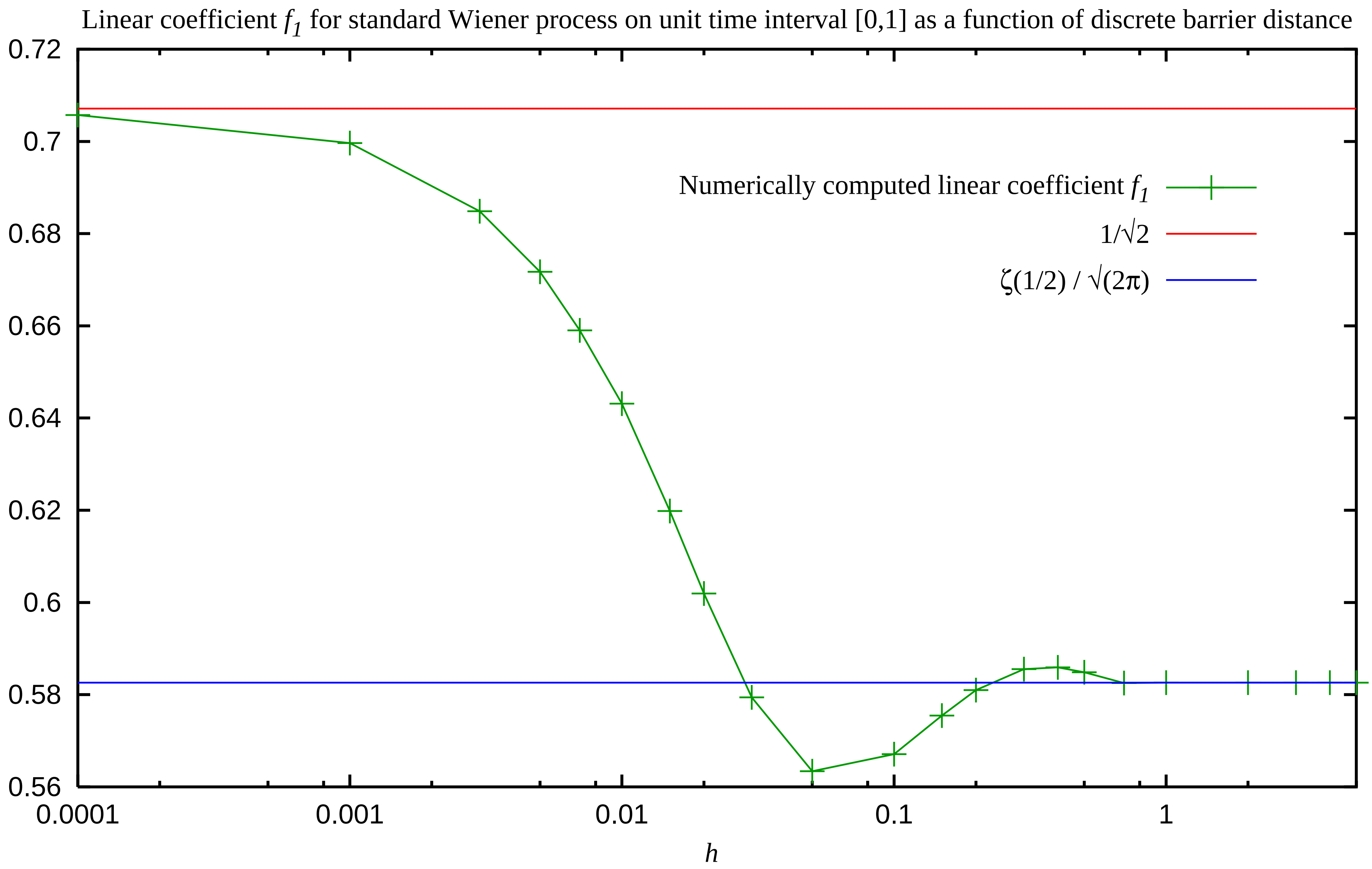
$$f_1(h_{\mathsf{d}}) \;=\; \lim_{\tau\to 0} \frac{h_c(h_{\mathsf{d}};\tau) - h_{\mathsf{d}}}{\sqrt{\tau}} \;=\; \lim_{\tau\to 0} \frac{\Phi^{-1}\left(\frac{1}{2} - \frac{1}{2}p_{\mathsf{survival}}(h_{\mathsf{d}};\tau)\right) - h_{\mathsf{d}}}{\sqrt{\tau}} \,. \tag{19}$$

The BGK correction formula gives us a boundary condition for f_1:

$$\lim_{x\to\infty} f_1(x) \;=\; \left|\zeta\left({}^1\!/\!{}_2\right)\right| / \sqrt{2\pi} \tag{20}$$

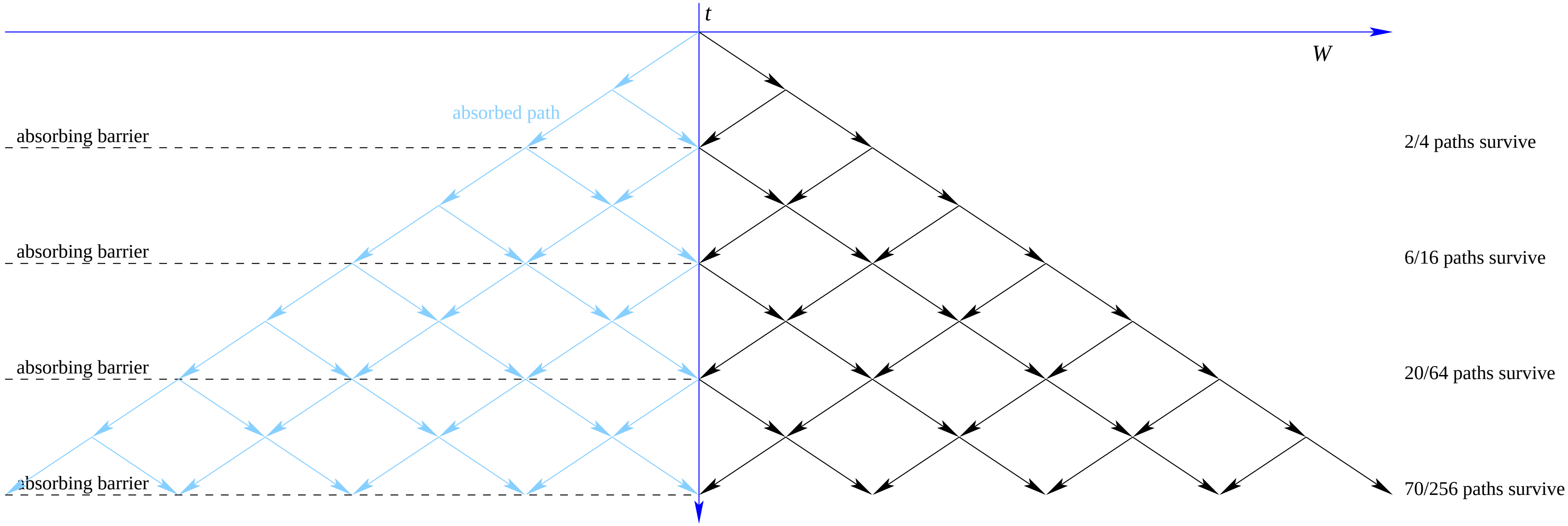
Using Fourier convolution and regression techniques, we can approximate $f_1(h_{\mathsf{d}})$ numerically also where the required assumptions for the BGK expansion no longer hold, i.e. near the barrier.

Understanding the behaviour near the barrier can be crucial for the successful hedge of a live derivatives position.


Linear coefficient f_1 for standard Wiener process on unit time interval [0,1] as a function of discrete barrier distance
0.72
0.7
0.68
0.66
0.64
0.62
0.6
0.58
0.56
0.0001
0.001
0.01
0.1
1
h
Numerically computed linear coefficient f_1
$1/\sqrt{2}$
$\zeta(1/2) / \sqrt{(2\pi)}$

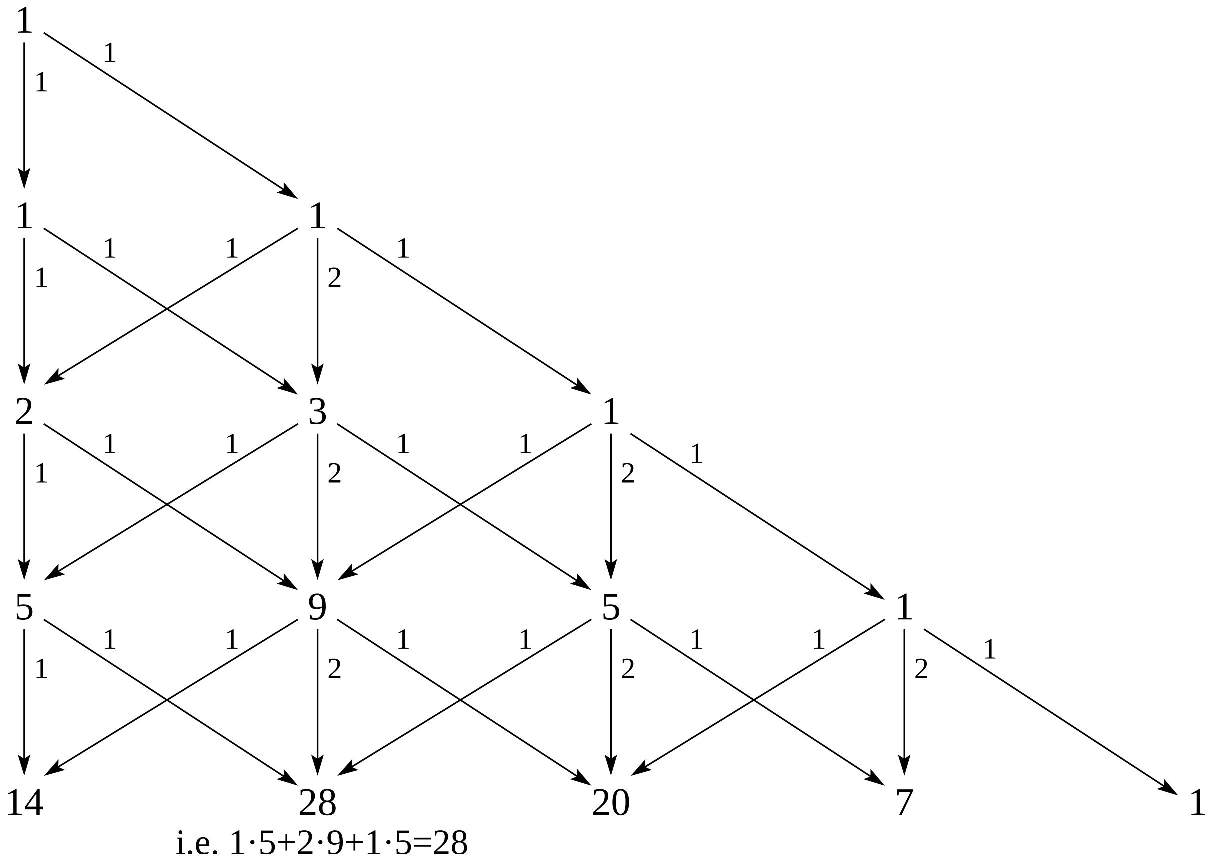
What's happening for $h_{\mathsf{d}} \to 0$? Let us examine the case $h_{\mathsf{d}} = 0$ more closely.

The absorption probability for n regularly spaced discrete absorption barriers that extend from 0 to $-\infty$ can be computed by the use of an equivalent binomial tree of $2n$ steps:



All paths that cross into the negative domain are considered absorbed.

This makes for an efficient counting algorithm effectively based on an explicit finite differencing implementation of the forward Kolmogorov equation:



Note: the so computed absorption probabilities are *exactly* equal to those resulting from a continuous process with n regularly spaced monitoring points.

With the row index $i = 0$ representing the start of the tree, and the column index $j = 0$ representing the node at the barrier, the number of paths that reach a node are given by the recursive rules

$$\begin{aligned} a_{i\,i} &= 1 \qquad \text{and} \qquad a_{i\,j} = 0 \;\; \forall \;\; j > i && (21) \\ a_{i\,0} &= a_{i-1\,0} + a_{i-1\,1} && (22) \\ a_{i\,j} &= a_{i-1\,j-1} + 2 \cdot a_{i-1\,j} + a_{i-1\,j+1} && (23) \end{aligned}$$

The probability of survival is[1]

$$p_{\text{survival}}(0, {}^1\!/\!_n) \;=\; \mathsf{E}\left[\prod_{k=1}^{n} \mathbf{1}_{\left\{\sum_{j=1}^{k} z_j > 0\right\}}\right] \;=\; 2^{-2n} \cdot \sum_{j=0}^{n} a_{n\,j} \qquad (24)$$

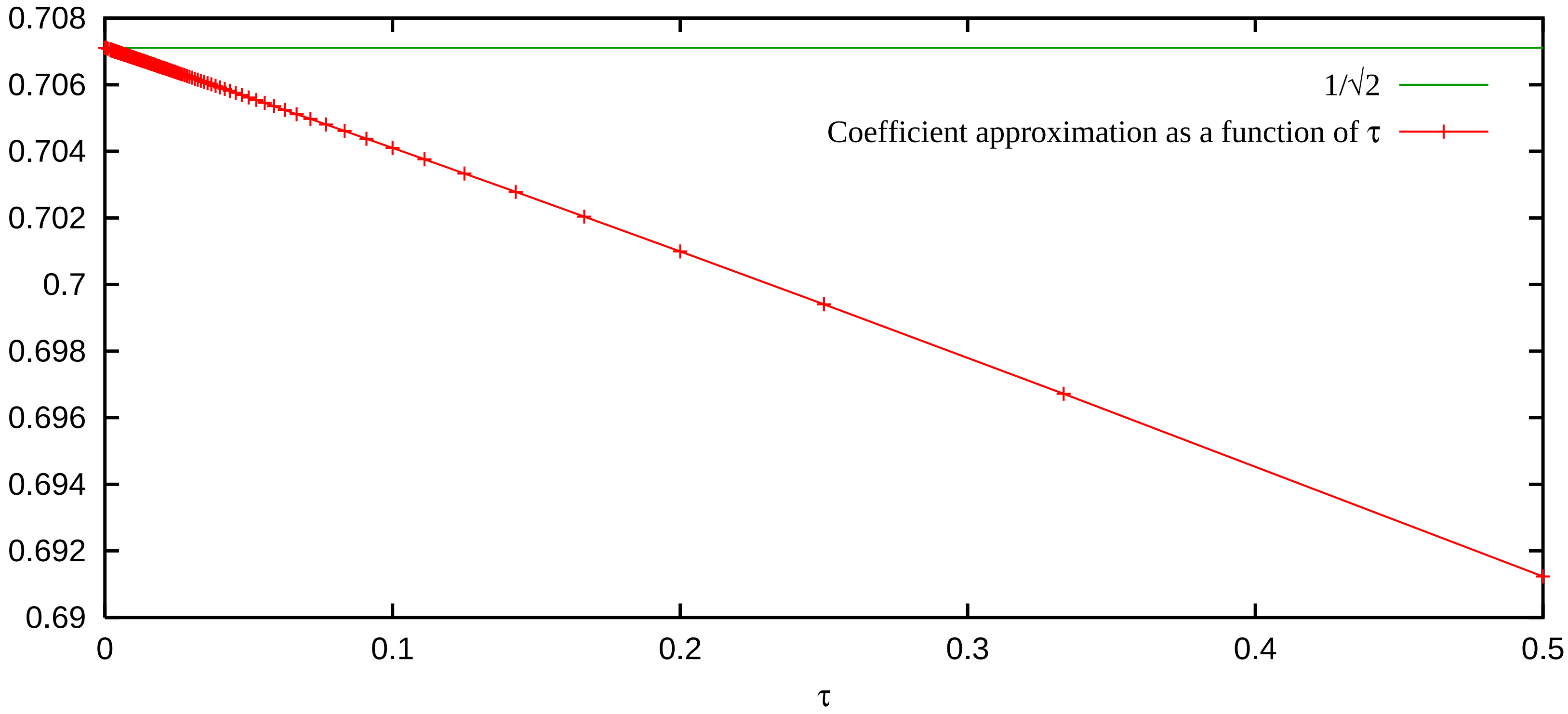
where all z_j are independent standard Gaussian variates.

[1]Unfortunately, I was not able to derive a closed form solution for $p_{\text{survival}}(0, {}^1\!/\!_n)$.

This enables us to compute the convergence of

$$f_1^{\text{approx}}(h_{\mathsf{d}};\tau)\Big|_{h_{\mathsf{d}}=0} = \frac{h_c(h_{\mathsf{d}};\tau)-h_{\mathsf{d}}}{\sqrt{\tau}}\Bigg|_{h_{\mathsf{d}}=0} = \frac{\Phi^{-1}\left(\frac{1}{2}-\frac{1}{2}p_{\text{survival}}(h_{\mathsf{d}};\tau)\right)-h_{\mathsf{d}}}{\sqrt{\tau}}\Bigg|_{h_{\mathsf{d}}=0} \tag{25}$$

for $\tau \to 0$ very accurately:



We obtain numerically

$$\begin{aligned}
\left(f_1^{\text{approx}}(0; 2^{-1}) \right)^2 &\approx 0.477804760 \\
\left(f_1^{\text{approx}}(0; 2^{-5}) \right)^2 &\approx 0.498690275 \\
\left(f_1^{\text{approx}}(0; 2^{-10}) \right)^2 &\approx 0.499959302 \\
\left(f_1^{\text{approx}}(0; 2^{-15}) \right)^2 &\approx 0.499998728
\end{aligned} \tag{26}$$

which means, on the scale of accuracy required for practical derivatives pricing purposes, we have $f_1(0) = \sqrt{{}^1\!/_2}$.

For the binomial tree

Conjecture: *The linear coefficient function $f_1(h_d)$ in the expansion (18) converges to $\sqrt{{}^1\!/_2}$ in the limit of $h_d \to 0$, i.e.*

$$\lim_{h_d \to 0} f_1(h_d) = \frac{1}{\sqrt{2}} \,. \tag{27}$$

Combining equation (16), conjecture (27), and the expansion of $\Phi(\epsilon)$

$$\Phi(\epsilon) \;=\; {}^1\!/\!_2 + \frac{\epsilon}{\sqrt{2\pi}} - \frac{\epsilon^3}{\sqrt{2\pi}} + \mathcal{O}(\epsilon^5)$$

has the following consequence:

The survival probability of a standard Wiener process on the unit time interval [0,1] with n regulary timed monitoring points at which the process is absorbed if it is below zero is given by

$$p_{\text{survival}}(0;1/n) \;=\; \frac{1}{\sqrt{n\pi}} + \mathcal{O}\left({}^1\!/\!_n\right) \;. \tag{28}$$

Numerically, we find

$$p_{\text{survival}}(0;1/n) \;\approx\; \frac{1}{\sqrt{n\pi}} \cdot \left(1 - \frac{1}{8n}\left(1 - \frac{1}{16n}\left(1 - \frac{5}{8n}\right)\right)\right) + \mathcal{O}\left(n^{-\frac{9}{2}}\right) \;. \tag{29}$$

Appendix

A. Transition density subject to double barrier knock-out

Define S as the initial spot value, L as a lower barrier and H as a higher barrier, and

h	z	l	
$\frac{\ln(H/S_t)}{\sigma\sqrt{\tau}}$	$\frac{\ln(S_{t+\tau}/S_t)}{\sigma\sqrt{\tau}}$	$\frac{\ln(L/S_t)}{\sigma\sqrt{\tau}}$	in the Black-Scholes (i.e. lognormal) model
$\frac{H-S_t}{\sigma\sqrt{\tau}}$	$\frac{S_{t+\tau}-S_t}{\sigma\sqrt{\tau}}$	$\frac{L-S_t}{\sigma\sqrt{\tau}}$	in the Bachelier (i.e. normal) model

as well as $\delta = h - l$ and

$$\alpha_{i1} = -2h - 2i\delta \quad \alpha_{i2} = -2(i+1)\delta \quad \alpha_{i3} = -2l + 2i\delta \quad \alpha_{i4} = 2(i+1)\delta \ .$$

Then, the transition density (assuming a drift-free process) over the time step τ subject to knock-out barriers at L and H can be derived using a recursive reflection principle to yield

$$\psi_{\text{drift-free transition}}(z) = \varphi(z) + \sum_{i=0}^{\infty}\sum_{j=1}^{4}(-1)^j\varphi(z+\alpha_{ij}) \ . \tag{30}$$

Alternatively, one can use a Fourier expansion approach to obtain the following expression for the same drift-free transition density

$$\psi_{\text{drift-free transition}}(z) \quad = \quad -\frac{2}{\delta}\sum_{j=0}^{\infty}\Bigg[\sin\left(\omega_{2j+1}\right)\mathrm{e}^{-\frac{1}{2}\omega_{2j+1}^2}\sin\left(\omega_{2j+1}(z-l)\right) + \sin\left(\omega_{2j+2}\right)\mathrm{e}^{-\frac{1}{2}\omega_{2j+2}^2}\sin\left(\omega_{2j+2}(z-l)\right)\Bigg] \tag{31}$$

with $\omega_j = \frac{j\pi}{\delta}$. Note that both in (30) and (31) terms are grouped to reduce the risk of spurious convergence if computation is continued until subsequent terms no longer contribute to the sum.

For constant process coefficients, the drift correction for the drift-free transition density is given by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \mathrm{e}^{\gamma z - \frac{\gamma^2}{2}} \qquad \text{with} \qquad \gamma = \begin{cases} \frac{(r-d-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} & \text{for Black-Scholes} \\ \frac{(r-d)T}{\sigma\sqrt{T}} & \text{for Bachelier} \end{cases} . \tag{32}$$

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