

The Gamma Loss and Prepayment model

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Abstract

We present¹ a model for the dynamics of fractional notional losses and prepayments on asset backed securities for the valuation and risk management of derivatives such as so-called *waterfall* structures [Colb] and other structured debt obligations on bespoke portfolios.

1 Introduction

In recent years, a variety of approaches have been proposed for the modelling of credit derivatives. Arguably, the emphasis of published models is on synthetic CDOs, and we give a brief overview of those models in this section before introducing the ideas needed for the modelling of asset backed securities and derivatives thereof.

The standard Gaussian copula model [Vas87, Li00] is consistent with a single-common-random-variable approximation of a multi-name structural distance-to-default model wherein the distance-to-default process is a weighted sum of a common and an idiosyncratic standard Brownian motion, and the default barrier is a function of time, adjusted such that individual marginal survival probabilities are recovered according to some given functions $p_i(T)$. When the codependence coefficient ρ is homogeneous across all names, individual survival probabilities conditional on a uniform common factor $u \in (0, 1)$ are identical to the Gaussian copula function

$$p_i(T)|_u = \Phi \left(\sqrt{1/\rho} \cdot \Phi^{-1}(p_i(T)) - \sqrt{\rho/1-\rho} \cdot \Phi^{-1}(1-u) \right). \quad (1)$$

Note that we have formulated the framework such that high values for the common factor, i.e. $u \nearrow 1$, give rise to increased survival probability, whereas low values of u , i.e. $u \searrow 0$, lead to increased risk of default. Conditional on the common factor, survival probabilities are independent, and thus, once the level of the common factor is known, independent default times can be drawn, or loss distributions can be built efficiently [RS03, ASB03, LG02]. Several authors have extended the copula framework, typically on the basis of a direct choice of an alternative copula [Bou00, HW05, Whe03], or by introducing adjustments that facilitate calibration to market information about loss codependence [AS04]. An exception is the work by Baxter [Bax06, Bax07] who extends the copula framework by reviewing the codependent structural default model framework, extending the distance-to-default process to allow for generic Lévy processes, and, specifically, investigating the Gamma process [DGS91]. A standard gamma process is a Lévy process with independent Gamma increments [App04]. The law of the increment Δx over time Δt is given by $\Gamma(\gamma \cdot \Delta t, \Delta x)$ with

$$\Gamma(a, x) = \int_0^x z^{a-1} e^{-z} dz / \Gamma(a) \quad (2)$$

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In the limit of what Baxter refers to as the *European approximation*, the choice of the Gamma process as a distance to default variable results in what may be called the *Gamma copula*

$$p_i(T)|_u = \Gamma((1 - \phi)\gamma, \Gamma^{-1}(\gamma, p_i(T)) - \Gamma^{-1}(\phi\gamma, 1 - u)) . \quad (3)$$

The inverse distribution function $\Gamma^{-1}(a, p)$ is defined as the solution of $p = \Gamma(a, x)$ for x such that $x = \Gamma^{-1}(a, \Gamma(\gamma, x))$. For $\gamma \rightarrow \infty$ the Gamma copula converges to the Gaussian copula. This is a feature it inherits from the associated Gamma process which, if centred and rescaled according to

$$x \rightarrow (x - \gamma t) / \sqrt{\gamma} , \quad (4)$$

converges to a standard Wiener process for the limit $\gamma \rightarrow \infty$.

Another stream of models is based on the concept of a stochastic conditional instantaneous default rate, conditional on the respective name not having defaulted yet. Early forerunners of this framework were articles by Duffie [Duf98] and Duffie, Pan, and Singleton [DPS00] in which the authors take advantage of analytical results available in an affine setting. Chapovsky, Rennie, and Tavares [CRT06] investigate a whole series of conditional instantaneous default processes that all have an idiosyncratic deterministic part, and a common stochastic component. They find that, in order to reflect market-realistic codependence structures, significant joint jumps in conditional instantaneous default rates are required. Lipton [Lip06] and Lipton and Inglis [LI07a, LI07b] address this requirement by setting his novel approach closer to a direct modelling of the conditional instantaneous default rate integral, a quantity more closely related to common-factor conditioned survival probabilities. They, too, require sizeable jumps for the conditional instantaneous default rate process to reflect market-implied codependence figures. Despite their differences, all of these approaches stay within the framework where any single obligor's default hazard rate, i.e. conditional instantaneous default rate, is a conventional jump-diffusion process with a deterministic drift term.

The observation that significant *joint* increases in the conditional hazard rate integral over short periods of time for several obligors are necessary to reflect the market view on codependence is a feature that is catered for in the more unusual setting of Joshi and Stacey [JS06]. They propose dynamics directly for the hazard rate integral, giving each obligor a conditional² survival probability equal to

$$E[\tau_i > T | I(\cdot)] = e^{-\int_0^T c_i(s) dI(s)} \quad (5)$$

wherein the common factor process $I(t)$ is a sum of Gamma processes³ and a deterministic drift. Joshi and Stacey proceed to describe how to generate common factor paths in a simulation context, and how to draw so conditioned default times for the valuation of portfolio credit derivatives.

Thus far, the focus for credit derivatives models appears to have been on models whose primary underlying concept is that of a *default event* of an underlying reference entity. Since most bespoke cashflow CDO structures based on asset backed securities typically incur *partial losses* in their underlyings, as opposed to complete default of the security, a different approach is needed when one wishes to design a model that relates the price of a cashflow CDO with its complicated nonlinear waterfall rules to observable prices of the respective underlying securities. The difference is that, for asset backed CDOs, we need to concern ourselves with the occurrence of repeated fractional losses on the underlying assets, as opposed to single default events on each of the underlying assets.

Inspired by the work of Baxter [Bax06] and Joshi and Stacey [JS06], we now provide details for a flexible portfolio credit derivatives modelling framework for asset backed securities. Like both Baxter and Joshi-Stacey, we use a Gamma process to drive the uncertainty. Like Joshi and Stacey, we use the gamma process to represent the conditional hazard rate integral, or *the negative logarithm of conditional survival probability*. Unlike Joshi and Stacey, we do not use multiple Gamma processes to represent the common factor, but instead

²conditional on the path of common factor processes

³For a brief introduction to gamma processes see [Fel71, RW00, DGS91, Bax06, JS06].

stay closer to Baxter’s way of correlating individual default processes. Unlike both Baxter and Joshi-Stacey, however, our primary concern is directly with the joint distribution of losses on the underlying assets. We therefore focus on the net effect that defaults of underlying obligors or collateralised assets have on the joint loss distribution, and in this manner, we take advantage of the ability to give up on the need for individual default times. This means that the modelling of the negative logarithm of conditional survival probabilities can also be interpreted as a direct modelling of the negative logarithm of proportional losses on the underlying asset. This view, and the observation that the underlying assets for asset backed portfolio derivatives are often sizeable pools of much smaller individual liabilities as, for instance, in the case of a mortgage pool, make the choice of a Gamma process all the more intuitive since the Gamma process is a model previously used for losses arising from a multitude of individual claims in actuarial and related sciences [PW84, DGS91].

2 The Gamma Loss and Prepayment model for asset backed securities

In the following, we assume that any asset backed derivative depends in its value and cashflow streams on a set of underlying securities. In the simplest case, such an underlying security is a bond backed by a pool of loans or mortgages. Within the model, we treat such underlying securities as a single asset whose notional can incur fractional erosion. In practice, it is also possible that the underlying securities are tranches from cash CDO transactions. When this is the case, we have two choices. Either, we identify the modelled asset as the loan pool behind the underlying cash CDO transaction, and incorporate the fact that the underlying is in fact a CDO tranche on this loan pool by means of explicitly coding up the CDO tranche payoff and redemption rules. Alternatively, one can approximate the underlying bond, even when it is in fact a cash CDO tranche, as a loan pool with its expected future conditional default and conditional prepayment rates if such data are available for calibration of the model. Both methodologies have their respectively arguable merits and downsides. Which of the two approaches is taken should in practice depend on what data are available for calibration and what instruments are available for (however limited) hedging.

2.1 The loss process

Without loss of generality, we normalize each underlying notional process $n_i(t)$ to be of unit size at inception, i.e. $n_i(0) = 1$. In the absence of prepayments, we assume that *losses* on the notional n_i are generated by the negative log-loss process $\xi_i(t)$ in the sense that the loss incurred between time t and $t + \tau$ is given by

$$\ell_i(t + \tau) - \ell_i(t) = n_i(t) \cdot \left(1 - e^{-(\xi_i(t+\tau) - \xi_i(t))}\right) . \quad (6)$$

The negative log-loss process $\xi_i(t)$ is represented in the model by a Gamma process with time-varying intensity:

$$\xi_i(T) = \int_0^T c_{x_i}(t) \cdot dx_i(t) . \quad (7)$$

In practice, we make the deterministic function $c_{x_i}(t)$ piecewise constant in time in aid of calibration to expected loss curves. The process $x_i(t)$ itself comprises two independent parts, namely

$$x_i(t) = \tilde{x}(\phi_x \cdot t) + \tilde{x}_i((1 - \phi_x) \cdot t) , \quad (8)$$

and this is where we follow Baxter’s methodology for the generation of codependence. The first part, \tilde{x} , is to be seen as the *common loss factor process*. Independent from it and from each other, but otherwise identical in all properties, we have one *idiosyncratic loss factor process* \tilde{x}_i for each of the underlying assets. All of the $\tilde{x}_i(s)$, and $\tilde{x}(s)$, are Gamma processes of expectation equal to their filtration time s , and variance equal to their filtration time multiplied by the square of the loss process volatility σ_x . A summary of these mathematical properties is given in equations (9) to (16).

$$x_i(t) = \tilde{x}(\phi_x \cdot t) + \tilde{x}_i((1 - \phi_x) \cdot t) \quad (9) \quad \mathbb{E}[\tilde{x}(t)] = t \quad (13)$$

$$\tilde{x}(t) \sim \psi_\gamma(\tilde{x}(t); t/\sigma_x^2, 1/\sigma_x^2) \quad (10) \quad \mathbb{V}[\tilde{x}(t)] = \sigma_x^2 \cdot t \quad (14)$$

$$\psi_\gamma(x; \alpha, \beta) = x^{\alpha-1} \beta^\alpha e^{-\beta x} / \Gamma(\alpha) \quad (11) \quad \text{Cov}[dx_i, dx_j] = \sigma_x^2 \cdot \phi_x \cdot dt \quad (15)$$

$$\mathbb{E}[e^{-\omega \cdot \tilde{x}(t)}] = (1 + \omega \cdot \sigma_x^2)^{-t/\sigma_x^2} \quad (12) \quad \text{Corr}[dx_i, dx_j] = \phi_x \cdot dt \quad (16)$$

We show examples for the density (11) in figure 1. Note that by virtue of $x_i(t)$ being a Gamma process, losses over any time interval dt are generally not infinitesimal. This is a desirable feature of the model

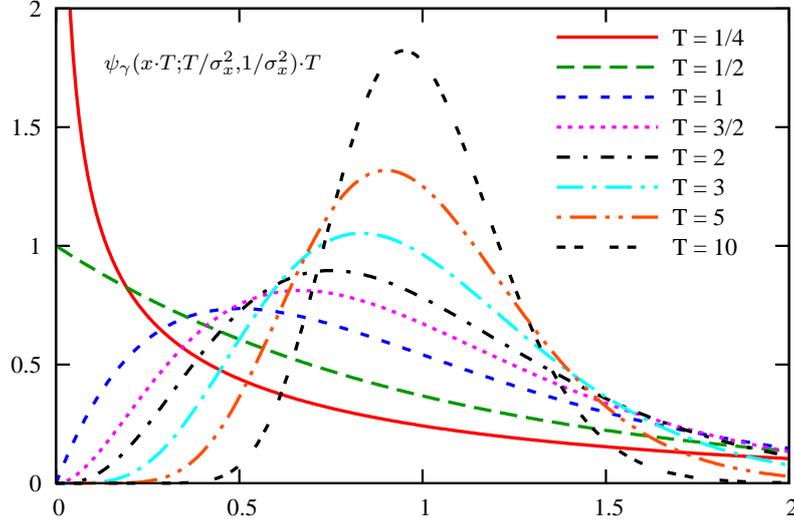


Figure 1: The Gamma density (11) for $\sigma_x = 1/\sqrt{2}$. The axes have been rescaled in order to compensate for the peak of the distribution drifting out linearly in T , i.e. the actual function shown is $\psi_\gamma(x \cdot T; T/\sigma_x^2, 1/\sigma_x^2) \cdot T$.

in order to capture sufficient codependence of losses consistent with market observables. It is exactly this feature of the cumulative loss process that makes the Gamma process preferable over any finite intensity model. For the Gamma Loss model, the closest equivalent to an instantaneous hazard rate is the derivative of $x_i(t)$ with respect to t . However, since a Gamma process is not only non-differentiable for all t (a trait it shares with standard Brownian motion), but also discontinuous at all points in time, as well as ever increasing, the concept of a temporal derivative is ill defined, and not necessarily helpful⁴. For the modelling of loss distributions, though, we do not depend on the existence of a meaningful hazard rate process, whence we dispose of the notion, and focus directly on losses. In the limit of $\sigma_x \rightarrow 0$, the process $x_i(t)$ converges to the deterministic process t which means that the model encompasses the limiting case of future losses happening exactly as forecast out of today.

2.2 The prepayment process

In addition to fractional notional losses due to default of individual constituents of underlying asset pools, we allow for notional reduction by means of prepayment. Within the model, prepayments are also represented by a Gamma process, in exactly the same fashion as losses, albeit with different parameters. In the absence of losses, the prepayment amount for underlying asset $\#i$ incurred between time t and $t + \tau$ is given by

$$\pi_i(t + \tau) - \pi_i(t) = n_i(t) \cdot (1 - e^{-(\eta_i(t+\tau) - \eta_i(t))}) \quad (17)$$

The negative log-prepayment process $\eta_i(t)$ is also defined as a Gamma process with time varying intensity according to

$$\eta_i(T) = \int_0^T c_{y_i}(t) \cdot dy_i(t) \quad (18)$$

and the dynamic characteristics of the negative log-prepayment process driver $y_i(t)$ are given in complete analogy to equations (9)–(16), with $x \rightarrow y$ in all occurrences. In addition, we assume

$$\text{Corr}[dx_i(t), dy_j(t)] = 0 \quad (19)$$

⁴This is in contrast to Brownian motion whose temporal derivative can still be interpreted as a meaningful process known as *white noise*. In fact, all diffusions can be seen as the temporal integral of respectively *coloured noise*.

i.e. we make all loss processes independent from all prepayment processes. This assumption is an obvious restriction of the presented model. Note, however, that independence between idiosyncratic incremental losses and idiosyncratic incremental prepayments does *not* mean that terminal losses and prepayments are independent. Naturally, if within one stochastic evolution of the model early losses are incurred, there is less notional available for later prepayment, and vice versa, which gives rise to a very natural loss and prepayment anti-correlation. In addition, it is not clear that a correlation between incremental losses and incremental prepayments is necessary from any economic principles. Historically, prepayments used to be considered dominated by interest rate moves since, to give but one economic argument, home owner's decisions for refinancing depend more directly on whether it is in their financial interest to do so than on whether their neighbours defaulted on their mortgage payments. In the interest of simplicity, and pending further evidence that correlation between incremental losses and incremental prepayments is strictly necessary for adequate modelling of asset backed securities, we proceed with the independence assumption.

2.3 The notional process

In the presence of both losses and prepayments, we define the notional process as

$$n_i(T) = e^{-(\xi_i(T) + \eta_i(T))} . \quad (20)$$

Equation (20) is the definition of the *Gamma Loss and Prepayment model*. By the aid of Itô's lemma for Lévy processes [CT04], which states that for a twice differentiable function $f(x, y)$ we have

$$\begin{aligned} df(x, y) &= \partial_x f(x, y) \cdot dx + \partial_y f(x, y) \cdot dy \\ &+ \mathbb{E}[f(x + dx, y + dy) - f(x, y) - \partial_x f(x, y) \cdot dx + \partial_y f(x, y) \cdot dy] , \end{aligned} \quad (21)$$

we can formally write the stochastic differential equation for the notional process n as

$$dn = -c_x \cdot n \cdot dx - c_y \cdot n \cdot dy + \mathbb{E}[ne^{-c_x \cdot dx - c_y \cdot dy} - n + c_x \cdot n \cdot dx + c_y \cdot n \cdot dy] \quad (22)$$

Using equation (12) and expanding

$$(1 + c_x \sigma_x^2)^{-dt/\sigma_x^2} = 1 - \ln(1 + c_x \sigma_x^2) / \sigma_x^2 \cdot dt + \mathcal{O}(dt^2) \quad (23)$$

we arrive at

$$\frac{dn}{n} = -c_x \cdot dx + (c_x - \ln(1 + c_x \sigma_x^2) / \sigma_x^2) \cdot dt - c_y \cdot dy + (c_y - \ln(1 + c_y \sigma_y^2) / \sigma_y^2) \cdot dt . \quad (24)$$

2.4 Calibration

For most asset backed securities, there is, alas, no liquid market for contracts on the respective securities. Instead, it has become common practice by rating agencies, data providers, and practitioners to think in terms of *conditional default rates* (CDR) and *conditional prepayment rates* (CPR) for any specific underlying asset backed security, and the Gamma Loss and Prepayment model is designed to allow easy calibration to such input data.

When prepayment and loss forecasts are given for a set of discrete time horizons T_j in the form of piecewise constant conditional deterministic instantaneous prepayment and loss rates $h_{x_{ij}}$ and $h_{y_{ij}}$, respectively, such that

$$\mathbb{E}[n_i(T_k)] = e^{-\sum_{j=1}^k (h_{x_{ij}} + h_{y_{ij}}) \cdot \tau_j} \quad (25)$$

with $\tau_j = T_j - T_{j-1}$ and $T_0 = 0$, the calibration curves $c_{x_i}(t)$ and $c_{y_i}(t)$ can be represented as piecewise constant functions of time whose levels are readily calibrated according to:-

$$c_{x_i}(t) = \left(e^{h_{x_{ij}} \cdot \sigma_x^2} - 1 \right) / \sigma_x^2 \quad \text{for} \quad T_{j-1} < t \leq T_j \quad (26)$$

$$c_{y_i}(t) = \left(e^{h_{y_{ij}} \cdot \sigma_y^2} - 1 \right) / \sigma_y^2 \quad \text{for} \quad T_{j-1} < t \leq T_j . \quad (27)$$

2.5 Loss and prepayment process disambiguation

The formulation (20) of the notional process alone does not suffice to resolve what losses are incurred by any one underlying asset (i.e. loan pool) over a given time step. However, due to ξ_i and η_i being independent Gamma processes, the notional process (20) permits the simple decomposition

$$\int_{s=0}^{\tau} dn_i(s) = \int_{s=0}^{\tau} e^{-\eta_i(s)} d(e^{-\xi_i(s)}) + \int_{s=0}^{\tau} e^{-\xi_i(s)} d(e^{-\eta_i(s)}) . \quad (28)$$

It is therefore intuitive to define the loss process

$$\ell_i(\tau) := - \int_{s=0}^{\tau} e^{-\eta_i(s)} d(e^{-\xi_i(s)}) \quad (29)$$

and the prepayment process

$$\pi_i(\tau) := - \int_{s=0}^{\tau} e^{-\xi_i(s)} d(e^{-\eta_i(s)}) . \quad (30)$$

Whilst of no further relevance in the following, one can also express these in differential notation,

$$d\ell_i(t) = - e^{-\eta_i(t)} d(e^{-\xi_i(t)}) = n_i(t) \cdot [c_{x_i} \cdot dx_i + (h_{x_i} - c_{x_i}) \cdot dt] \quad (31)$$

$$d\pi_i(t) = - e^{-\xi_i(t)} d(e^{-\eta_i(t)}) = n_i(t) \cdot [c_{y_i} \cdot dy_i + (h_{y_i} - c_{y_i}) \cdot dt] , \quad (32)$$

wherein we have used the calibration conditions (26) and (27). In the case of $\sigma_x \rightarrow 0$ and $\sigma_y \rightarrow 0$, the stochastic differential equations (31) and (32) reduce to

$$d\bar{\ell}_i(t) = \bar{n}_i(t) \cdot c_{x_i}(t) \cdot dt \quad (33)$$

$$d\bar{\pi}_i(t) = \bar{n}_i(t) \cdot c_{y_i}(t) \cdot dt \quad (34)$$

wherein the bar indicates that this is specific to the zero volatility case. The solutions to equations (33) and (34) are

$$\bar{\ell}_i(t + \tau) - \bar{\ell}_i(t) = n_i(t) \cdot \frac{h_{x_i}}{h_{x_i} + h_{y_i}} \cdot (1 - e^{-(h_{x_i} + h_{y_i})\tau}) \quad (35)$$

$$\bar{\pi}_i(t + \tau) - \bar{\pi}_i(t) = n_i(t) \cdot \frac{h_{y_i}}{h_{x_i} + h_{y_i}} \cdot (1 - e^{-(h_{x_i} + h_{y_i})\tau}) , \quad (36)$$

exactly as we expect for deterministic loss and prepayment rates. In expectation, the loss process increases according to

$$\begin{aligned} \mathbb{E}[\ell_i(\tau)] &= - \mathbb{E} \left[\int_{t=0}^{\tau} e^{-\eta_i(t)} d(e^{-\xi_i(t)}) \right] \\ &= - \int_{t=0}^{\tau} \mathbb{E} [e^{-\eta_i(t)}] \mathbb{E} [d(e^{-\xi_i(t)})] \\ &= - \int_{s=0}^{\tau} e^{-\int_{s=0}^t h_{y_i}(s) ds} d \left(e^{-\int_{s=0}^t h_{x_i}(s) ds} \right) \\ \mathbb{E}[\ell_i(\tau)] &= \int_{t=0}^{\tau} h_{x_i}(t) \cdot e^{-\int_{s=0}^t (h_{x_i}(s) + h_{y_i}(s)) ds} dt , \end{aligned} \quad (37)$$

where we again substituted the calibration conditions (26) and (27) where needed. Equally,

$$\mathbb{E}[\pi_i(\tau)] = \int_{s=0}^{\tau} h_{y_i}(t) \cdot e^{-\int_{s=0}^t (h_{x_i}(s) + h_{y_i}(s)) ds} dt , \quad (38)$$

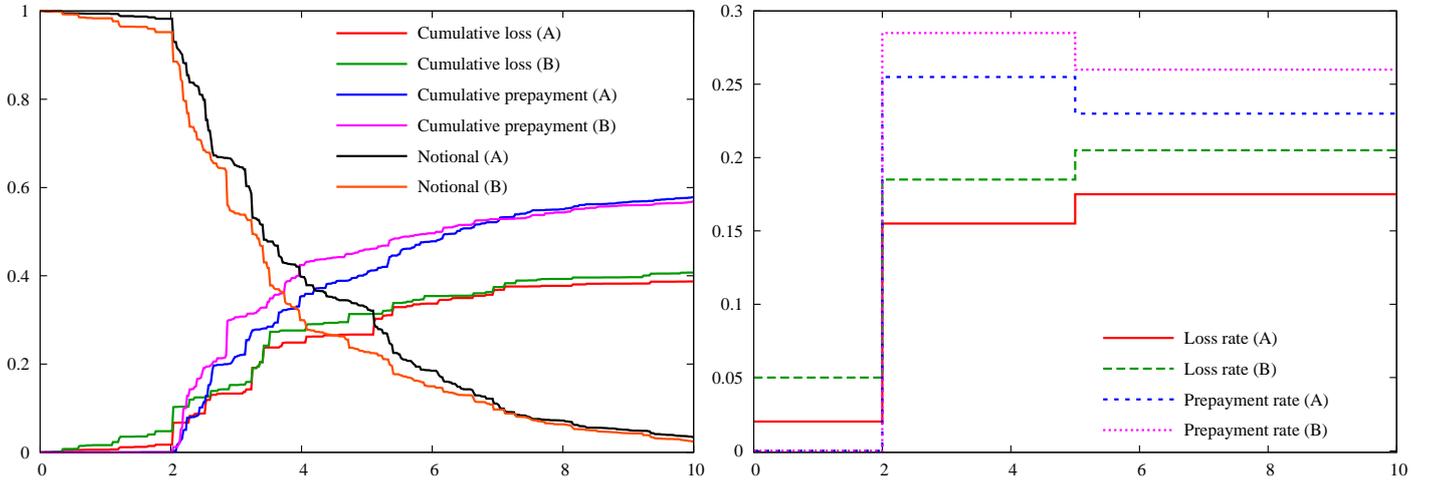


Figure 2: Sample paths for codependent cumulative losses, prepayments, and notionals (left), and associated expected loss and prepayment rates h_{x_i} and h_{y_i} (right).

which means that the loss and prepayment processes, as defined in (29) and (30), *in expectation*, evolve exactly as they do in the zero volatility case.

As a last point in this section, in order to permit the reader a visual impression of the dynamical behaviour of the Gamma Loss and Prepayment model, we show in figure 2 some sample paths for the codependent losses, prepayments, and notionals for two underlying assets A and B with loss correlation $\phi_x = 60\%$, loss volatility $\sigma_x = 70\%$, prepayment correlation $\phi_y = 40\%$, and prepayment volatility $\sigma_y = 30\%$ over a ten year horizon.

2.6 The loss density

As with any other credit model, it is of great practical use to have an understanding as to what loss distributions the respective analytics induce. For the Gamma Loss and Prepayment model, we first demonstrate the loss distribution for a single underlying in the absence of prepayments. Straightforward derivation yields for the density of the loss ℓ at time T

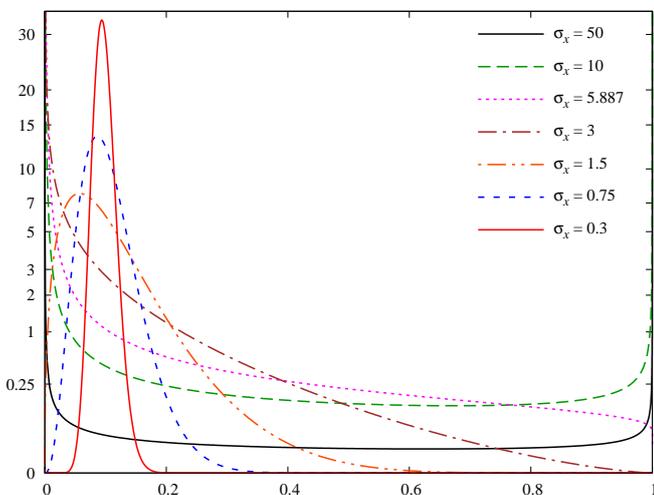
$$\psi(\ell) = \psi_\gamma \left(x; \frac{T}{\sigma_x^2}, \frac{1}{\sigma_x^2} \right) \cdot \frac{dx}{d\ell} = \psi_\gamma \left(-\frac{\ln(1-\ell)}{c_x}; \frac{T}{\sigma_x^2}, \frac{1}{\sigma_x^2} \right) / (c_x \cdot (1-\ell)) \quad (39)$$

where we have assumed constant h_x (and thus c_x), and $\psi_\gamma(\cdot; \cdot, \cdot)$ and c_x are given in equations (11) and (26), respectively. We show some sample curves in figure 3. We see that for $\sigma_x \rightarrow 0$, the loss density takes the shape of a Dirac function at $1 - e^{-h_x \cdot T}$, and for $\sigma_x \rightarrow \infty$, the density splits into two Dirac spikes, one at zero, and one at one, with weights $e^{-h_x \cdot T}$ and $1 - e^{-h_x \cdot T}$, respectively.

The latter scenario is of course exactly what we expect from a conventional Poisson-intensity model where the dynamics are given by a single and complete default event whose arrival probability over any time interval dt is always $h_x dt$. A useful result for the Gamma Loss model is that we can compute the critical volatility σ_x^* beyond which the density splits into a lower and an upper part, i.e. beyond which the loss density has a minimum. The critical volatility is

$$\sigma_x^* = \sqrt{\ln 2} / \sqrt{h_x}, \quad (40)$$

Figure 3: Loss densities for a single underlying with $T = 5$, $h_x = 2\%$, $h_y = 0$ and different values for σ_x .



which, for the example in figure 3, is 5.887. With respect to the loss distribution on portfolios of underlying

ings, we give some examples in figures 4 to 7. Note that the ordinate is on a cubic root scale throughout figures 3 to 7 in aid of making both low and high density values distinguishable in the same figure. The signature of the discrete nature arising from a binomial distribution can be seen clearly for the case of $\sigma_x = 50$.

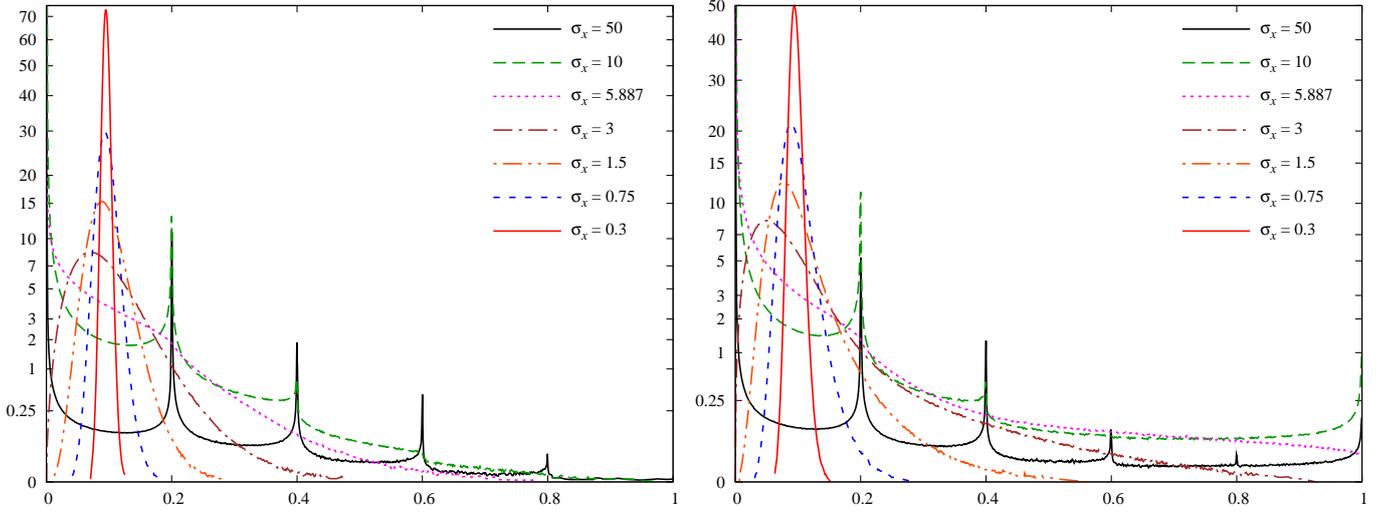


Figure 4: Loss densities for a portfolio of five equal underlyings with $T = 5$, $h_x = 2\%$, $h_y = 0$, and different values for σ_x as stated in the legend for $\phi_x = 0$ (left) and $\phi_x = 30\%$ (right).

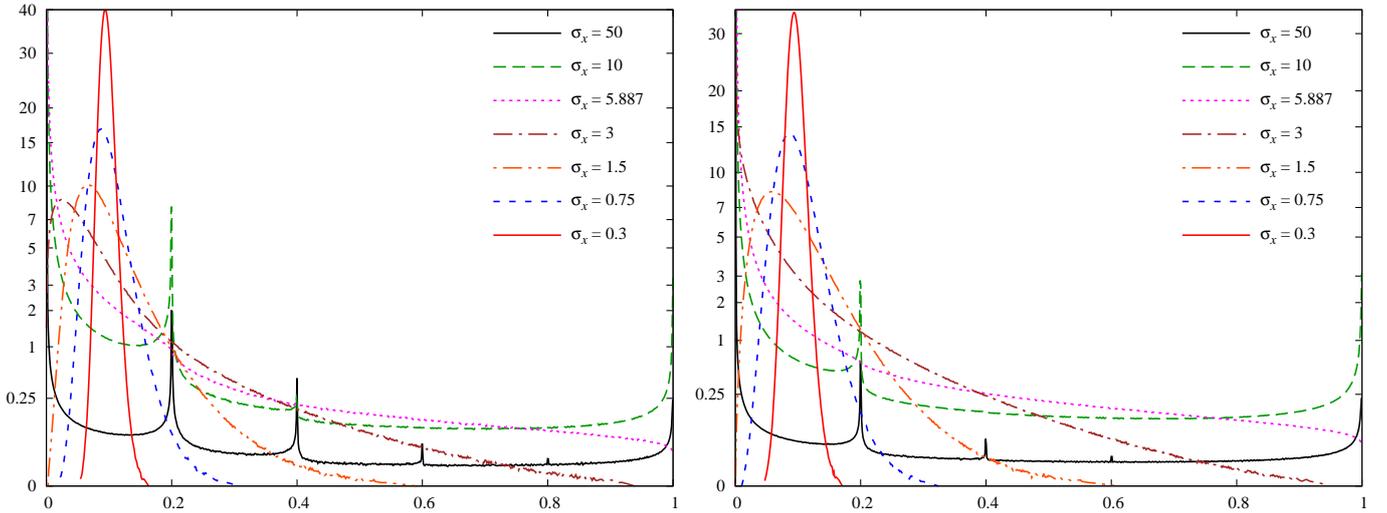


Figure 5: Loss densities for a portfolio of five equal underlyings with $T = 5$, $h_x = 2\%$, $h_y = 0$, and different values for σ_x as stated in the legend for $\phi_x = 60\%$ (left) and $\phi_x = 90\%$ (right).

3 Implementation

Numerical evaluation of complex derivative contracts such as waterfall structures is to be done by the aid of Monte Carlo simulations, which for the respective products typically requires monthly time steps. The Gamma processes can be constructed from incremental Gamma variate draws that can be generated, for instance, by the aid of the inverse cumulative Gamma distribution for which efficient algorithms are available [DM87].

As for the simulation of the loss and prepayment processes as defined in (29) and (30), we use a numerical integration scheme to approximate terms of the type

$$\Delta l_i := l_i(t + \tau) - l_i(t) \tag{41}$$

$$= n_i(t) \cdot \int_{s=t}^{t+\tau} e^{-c_{y_i} \int_{u=0}^s dy_i(u)} d \left(e^{-c_{x_i} \int_{u=0}^s dx_i(u)} \right), \tag{42}$$

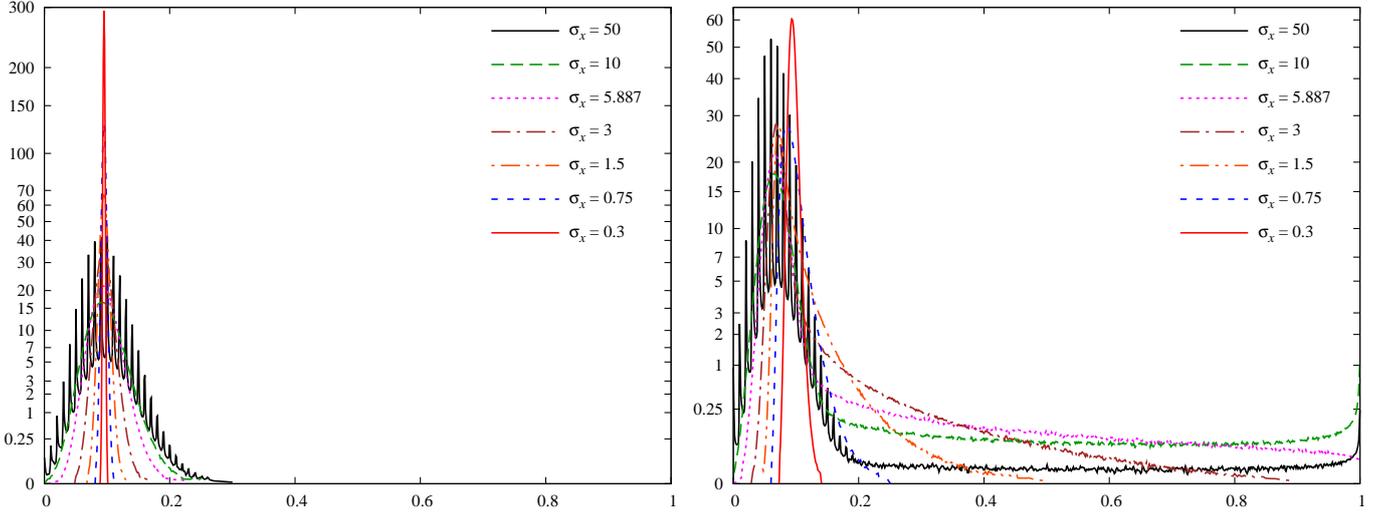


Figure 6: Loss densities for a portfolio of one hundred equal underlyings with $T = 5$, $h_x = 2\%$, $h_y = 0$, and different values for σ_x as stated in the legend for $\phi_x = 0$ (left) and $\phi_x = 30\%$ (right).

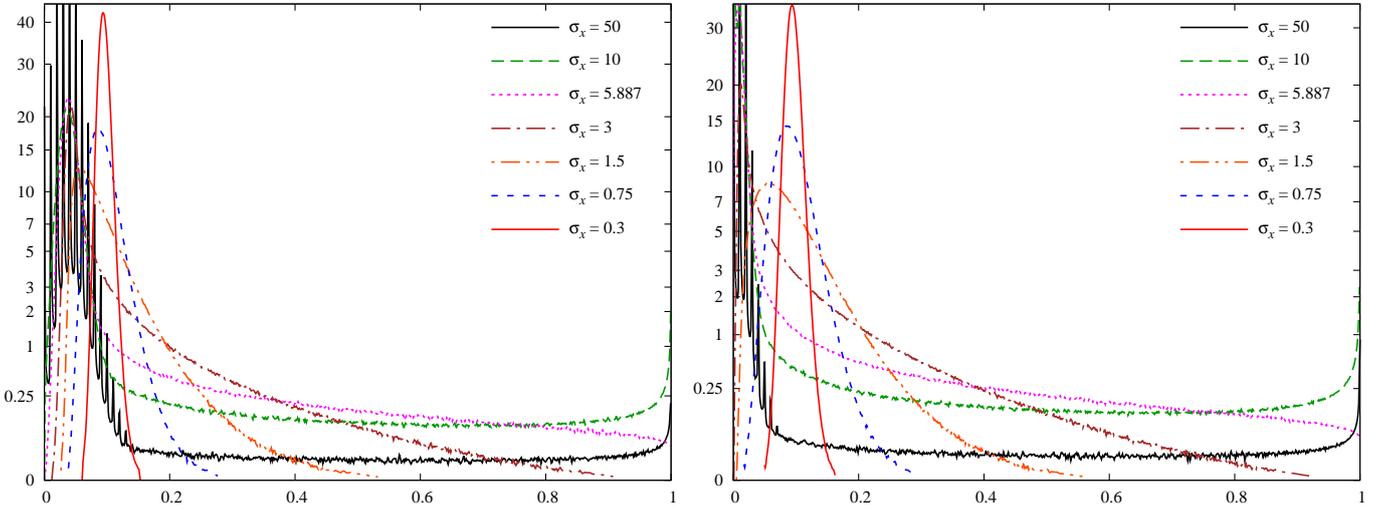


Figure 7: Loss densities for a portfolio of one hundred equal underlyings with $T = 5$, $h_x = 2\%$, $h_y = 0$, and different values for σ_x as stated in the legend for $\phi_x = 60\%$ (left) and $\phi_x = 90\%$ (right).

assuming that c_{x_i} and c_{y_i} are constant over the time step. Three different schemes present themselves:-

- Bias-order 1:

$$\Delta \ell_i^{[1]} := n_i(t) \cdot 1 \cdot (1 - e^{-c_{x_i} \cdot \Delta x_i}) \quad (43)$$

with

$$\Delta x_i = x_i(t + \tau) - x_i(t). \quad (44)$$

This is a standard explicit Euler scheme whose relative upwards bias is given by

$$\frac{\mathbb{E}[\Delta \ell_i^{[1]}]}{\mathbb{E}[\Delta \ell_i]} - 1 = \frac{1}{2} \cdot h_{y_i} \tau + \mathcal{O}(\tau^2) \quad (45)$$

- Bias-order 2:

$$\Delta \ell_i^{[2]} := n_i(t) \cdot \frac{1}{2} (1 + e^{-c_{y_i} \cdot \Delta y_i}) \cdot (1 - e^{-c_{x_i} \cdot \Delta x_i}) \quad (46)$$

This scheme can be seen as an Euler scheme with the integrand value being approximated by its own temporal average over the time step. The integrand's average is itself approximated by a simple trapezoidal rule. The relative bias of this scheme is

$$\frac{\mathbb{E}[\Delta \ell_i^{[2]}]}{\mathbb{E}[\Delta \ell_i]} - 1 = \frac{1}{12} \cdot h_{y_i} (h_{y_i} - h_{x_i}) \tau^2 + \mathcal{O}(\tau^4) \quad (47)$$

• Bias-order 3:

$$\Delta \ell_i^{[3]} := n_i(t) \cdot \left(a_0 + a_1 \cdot e^{-c_{y_i} \cdot \Delta y_i} + a_2 \cdot (e^{-c_{y_i} \cdot \Delta y_i})^2 \right) \cdot (1 - e^{-c_{x_i} \cdot \Delta x_i}) \quad (48)$$

with

$$a_0 := \frac{1}{2} + \frac{1}{6} \cdot \frac{h_{x_i} - h_{y_i}}{g_{y_i}} \quad g_{y_i} := \ln \left(2e^{h_{y_i} \sigma_y^2} - 1 \right) / \sigma_y^2 \quad (49)$$

$$a_1 := \frac{1}{2} + \frac{1}{6} \cdot \frac{h_{x_i} - h_{y_i}}{h_{y_i} - g_{y_i}} \quad a_2 := \frac{1}{6} \cdot \frac{h_{y_i}}{g_{y_i}} \cdot \frac{h_{y_i} - h_{x_i}}{h_{y_i} - g_{y_i}} \quad (50)$$

The relative bias of this scheme is

$$\frac{\mathbb{E}[\Delta \ell_i^{[3]}]}{\mathbb{E}[\Delta \ell_i]} - 1 = \frac{1}{72} h_{y_i} (h_{x_i} - h_{y_i}) (h_{y_i} - 2g_{y_i}) \tau^3 + \mathcal{O}(\tau^4) \quad (51)$$

We show the numerical magnitude of the relative bias from exact calculations (not just the leading order error term) for all three in figure 8. Naturally, integration schemes for the prepayment process follow in complete

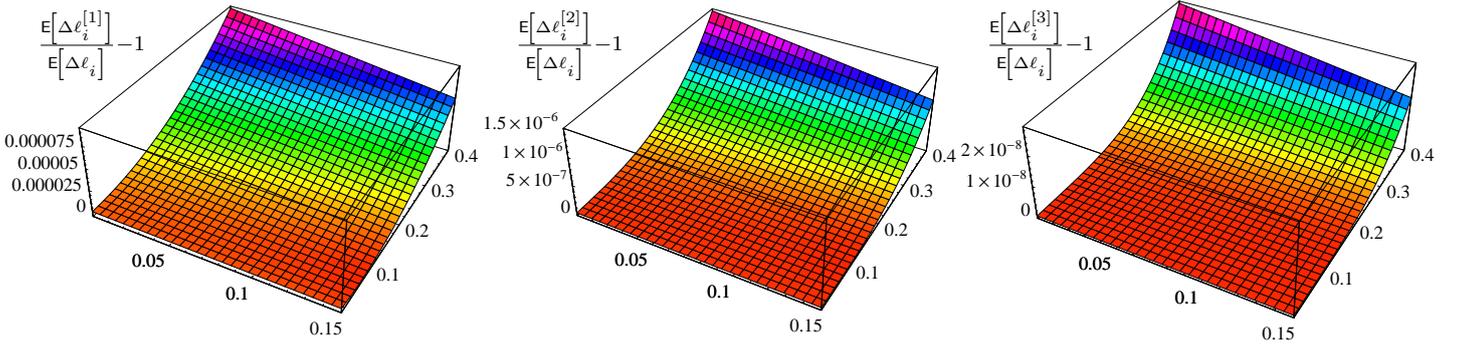


Figure 8: The relative bias of schemes (43) [left], (46) [middle], and (48) [right] for $\sigma_x = 70\%$, $\sigma_y = 40\%$, $\tau = 1/12$ on $h_x \in [0, 0.15] \times h_y \in [0, 0.4]$.

analogy to the loss process. Out of these schemes, in practice, we prefer the bias-order 2 scheme (46). This is because, whilst its bias order is inferior to that given in (48), it has the added advantage that loss increments and prepayment increments add up *exactly* to the notional decay:

$$\Delta \ell_i^{[2]} + \Delta \pi_i^{[2]} = \frac{n_i(t)}{2} \cdot \left[(1 + e^{-c_{y_i} \cdot \Delta y_i}) \cdot (1 - e^{-c_{x_i} \cdot \Delta x_i}) + (1 + e^{-c_{x_i} \cdot \Delta x_i}) \cdot (1 - e^{-c_{y_i} \cdot \Delta y_i}) \right] \quad (52)$$

$$= \frac{n_i(t)}{2} \cdot \left[1 + e^{-c_{y_i} \cdot \Delta y_i} - e^{-c_{x_i} \cdot \Delta x_i} - e^{-c_{y_i} \cdot \Delta y_i - c_{x_i} \cdot \Delta x_i} + 1 + e^{-c_{x_i} \cdot \Delta x_i} - e^{-c_{y_i} \cdot \Delta y_i} - e^{-c_{x_i} \cdot \Delta x_i - c_{y_i} \cdot \Delta y_i} \right] \quad (53)$$

$$= n_i(t) \cdot [1 - e^{-c_{x_i} \cdot \Delta x_i - c_{y_i} \cdot \Delta y_i}] \quad (54)$$

$$= n_i(t) - n_i(t + \tau) . \quad (55)$$

Given that the bias of scheme (46) is of negligible magnitude, the benefit of preserving the sum of losses and prepayments outweighs that of a higher bias convergence order.

4 Concluding remarks

We have presented a model for the representation of the most relevant risk factors in asset backed derivatives such as *waterfalls*, *cash CDOs*, and other complex structured debt contracts. The general idea is that we assume that the derivative contract is based on a set of asset backed bonds, such as subprime mortgage bonds. Within the model, we use one pair of underlying processes for each of the underlying asset backed bonds' loan pool. The two processes are a simple macroscopic representation of the losses and prepayments on the respective underlying liabilities, which are assumed to be an amorphous pool of defaultable debt. The model's strengths are that it readily calibrates to the market convention of conditional (future) default rates (CDR) and conditional (future) prepayment rates (CPR). It offers a single control parameter for the level of stochasticity in losses and prepayments, respectively, given by the volatility figures σ_x and σ_y . It also allows control over the codependence strength between all losses, and all prepayments, though independently between instantaneous *incremental* losses and prepayments. *Cumulative* losses and prepayments, however, are anti-correlated in this model by virtue of the natural mechanism of depletion since any scenario in which a lot of losses arise (in absolute terms) leaves little notional available for prepayments.

There are clearly many aspects that we have not touched upon with this model, such as calibration to the few market observable tranches (e.g. TABX), though, this is in our opinion of lesser importance since the model is primarily designed for complex structures on bespoke portfolios. We do feel, however, that many of the bespoke structures that are traded in the asset backed security market are amenable to at least an approximate evaluation with this model, given some compromises on the representation of some of the specific features. Realized recovery rates from defaults, for instance, can be approximated, if need be, as an increased prepayment rate. Whilst this is clearly not exact within this model, it is an adjustment that caters for most of the effect of recovery, and keeps the overall modelling framework simple enough to be workable. Despite all these obvious shortcomings of the model, we have, however, been able to calibrate the model to all relevant data we had available for all the structures we have analyzed with it to date.

As for an example for a real bespoke waterfall structure, any practitioner in this area will confirm that the description of such contracts is invariably extremely lengthy⁵, and thus naturally beyond the scope of this article, though we suggest some references for more detailed information [Ade, Colb, Cola, Tav03]. Not surprisingly, the algorithmic implementation of all the payoff rules, is a major effort, whence, historically, it used to be common practice to use simplifications to the payoff rules, and deterministic cash flow forecast scenarios in any attempt of valuation. As is clearly beyond any doubt at this point in the history of finance, the consequences of oversimplification can be disastrous since the risks contained in complex contingent contracts on asset backed debt pools are easily grossly underestimated when layers of simplifications are put together. Therefore, we considered it prudent to invest in the implementation of the full payoff structure of a given waterfall deal, in order to be able to have a deeper understanding of the loss distributions that may result from the complex payoff rules. Since asset backed debt derivatives tend to be bespoke, but with common features, we used a product description language for the implementation of the cashflow rules, and kept the layout as modular as possible, in order to allow for the flexibility needed to cater for alternative bespoke deals. It is, in our opinion, conceivable that this investment in modelling and technology can be key to a revival of the asset backed derivatives market.

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⁵Term sheets are typically more than one hundred printed pages, and often more than two hundred.

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