The Future is Convex

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Abstract

We present analytical approximation formulæ for the price of interest rate futures contracts derived from the yield curve dynamics prescribed by a Libor market model allowing for an implied volatility skew generated by displaced diffusion equations. The derivation of the formulæ by the aid of Itô-Taylor expansions and heuristic truncations and transformations is shown, and the results are tested against numerical calculations for a variety of market parameter scenarios. The new futures convexity formulæ are found to be highly accurate for all relevant market conditions, and can thus be used as part of yield curve stripping algorithms.

1 Introduction

Almost all interest rate derivatives modelling starts with a procedure practitioners often refer to as the *stripping of the yield curve*. The details can be rather involved, and most major investment houses have their own, possibly secret, methods for the interpolation of discount factors or overnight forward rates, and techniques how to incorporate the current prices of all relevant quoted market instruments such as short term cash deposits, exchange traded futures on interest rates, swap rates, and so on. Extreme care is used in the implementation of the bond and derivatives pricing system with respect to the precise handling of all involved rollover conventions, market centre holiday information, and other trading specifics that can make life rather interesting indeed for those who have to maintain the yield curve construction libraries. Having said all of the above, there is one exception to the rule of high precision with respect to all the numerical details in the yield curve construction: the integration of futures quotes into the set of yield curve instruments is usually done by the aid of an ad-hoc adjustment or approximate convexity corrections. These days, all practitioners know that futures quotations are not to be taken as a fair forward rate value due to an effect known as *convexity*. In a nutshell, one expects interest rate futures to have a higher value than the associated forward rate. This is because, when rates rise, the net present value of a long position in a forward rate contract rises in value but not as much as the forward rate itself since, as long as the whole yield curve is positively correlated, there tends to be more of a discounting effect when the forward rate has gone up. In contrast, the equivalent futures position pays the rise in value immediately, thus not suffering the effect of (the increased) discounting. Whilst this is simplistic, it makes it immediately plausible that futures quotes may differ from forward rates, and that this difference may depend on the volatility and correlation of different interest rates.

A very common approach to adjust futures quotes such that they can be used as forward rates in the yield curve stripping procedure is to estimate the convexity effect using a very simple, usually one factor, interest rate model with very approximate input numbers. Naturally, the very formula one has to apply depends on the actual model used for this purpose. For the extended Vasicek (also known as Hull-White) [HW90] and Cox-Ingersoll-Ross [CIR85] model, explicit formulæ can be derived. The situation is different for models whose continuous description gives the short rate a lognormal distribution such as the Black-Derman-Toy [BDT90] and Black-Karasinski [BK91] models: for these, in their analytical form of continuous evolution, futures prices can be shown to be positively infinite [HJM92, SS94]. However, as has been known by practitioners for some time ([Reb02], page 13), any time-discretised

approximation to these models does *not* incur these explosions[1]. Since no closed form solutions are available for the pricing of the traded securities the models must be calibrated to, the BDT and Black-Karasinski models were only ever implemented and calibrated (both to the yield curve and to volatility dependent derivatives such as caps or bond options) as a numerical scheme, and so the potential issue of exploding futures prices for lognormal short rate models gradually came to be ignored by most quantitatie analysts. An interesting study of the impact of model choice on the magnitude of the resulting futures price is the article by Gupta and Subrahmanyam [GS00]. They find a mild dependence of futures convexities on the choice of model for maturities up to five years, and this has grown to be the consensus amongst many quants: when comparing like-for-like calibrated models of different dynamics, the actual choice of model has a small influence on the magnitude of the convexity correction implied.

In contrast to short rate models and those that are derived from the continuous HJM framework, little work has been published on futures convexity corrections in the framework of (Libor or swap rate) market models. Matsumoto [Mat01] derived an approximation for futures prices for the standard Libor market model that is based on a first order expansion (in forward rate covariances) of the Radon-Nikodym derivative used for the transformation from spot to forward measure with the additional assumption that the resulting sum of weighted (lowest order drift-adjusted) forward rates is well represented as a basket of lognormal variates, alas without any numerical experiments for comparison. Our own experience, though, is that at least second order terms are required with this method in order to obtain a satisfactorily accurate approximation of the target distributions, and the aim of this article is to present a different approach including higher order terms leading to a level of accuracy that is satisfactory in all realistic market scenarios.

To date, probably the most frequently employed formulæ for futures convexity approximations are the results published by Kirikos and Novak in 1997 [KN97] for the extended Vasicek model of Gaussian short rates. The proliferation of this formula is so wide-spread that it is by now almost universally used for the purpose of yield curve stripping, and we, too, had become used to relying on just one convexity correction methodology. We were very surprised, then, when we recently tested our standard Hull-White convexity correction formulæ against the numbers produced by a numerically evaluated futures contract computed with the aid of a fully calibrated Libor market model in the lately more and more important case of low interest rates with comparatively high (relative) volatilities: we had to realise that the convexity correction of the Libor market model could easily be twice what we thought it would roughly be. Several factors are coming together to topple the assumed weak model dependence of futures convexity corrections: interest rates in JPY and USD are low, (Black) implied volatilities of caplets are high, and futures are more and more frequently reasonably liquid for expiries that go well beyond the hitherto assumed threshold of around two years. In fact, for the USD market, we find that we can deal in futures quite readily up to five years, and as we will elaborate in this article, the process and distributional assumptions of the chosen model give rise to noticeably different sizes of futures' convexity at such maturities. The specific model we have used for our approximations and numerical results presented here is the Libor market model [BGM97, Jam97, MSS97], but we also provide comparison with the conventional correction as derived by Kirikos and Novak.

The method we employ to arrive at our formulæ is a combination of formal Itô-Taylor expansions, selection of dominant terms, and heuristic adjustments of the resulting approximations preserving asymptotic equality (in the limit of vanishing variance), in order to arrive at manageable, yet sufficiently accurate expressions. Our approach is thus not dissimilar to that of [HKL02] used for a different purpose, namely to derive implied volatility approximations for a stochastic volatility process of the underlying, albeit that we cannot compare with the elegance presented there.

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[1]Unless, of course, extremely high volatilities are used in conjunction with a numerical scheme that is not unconditionally stable.

2 Brief review of the Libor market model

In the Libor market model for discretely compounded interest rates, we assume that each of a set of spanning forward rates f_i evolves lognormally according to the stochastic differential equation

$$\frac{\mathrm{d}f_i}{f_i} = \mu_i(\boldsymbol{f}, t)\,\mathrm{d}t + \sigma_i(t)\,\mathrm{d}\widetilde{W}_i \ . \tag{1}$$

From here on, we will rely on the convention that the volatility functions $\sigma_i(t)$ drop to zero after the fixing time of their associated forward rates since this facilitates the notation in our resulting integral formulæ. Correlation is incorporated by the fact that the individual standard Wiener processes in equation (1) satisfy

$$\mathsf{E}\left[\mathrm{d}\widetilde{W}_i\,\mathrm{d}\widetilde{W}_j\right] = \varrho_{ij}\mathrm{d}t \ . \tag{2}$$

If a zero coupon bond that pays one currency unit at t_N is used as numéraire, then the drift μ_i in equation (1) associated with the forward rate f_i that fixes at time t_i and pays at time t_{i+1} is given by:

$$\mu_i^{(t_N)}(\boldsymbol{f}(t), t) = -\sigma_i(t)\underbrace{\sum_{k=i+1}^{N-1} \frac{f_k(t)\tau_k}{1 + f_k(t)\tau_k}\sigma_k(t)\varrho_{ik}}_{\text{non-zero for } i < N-1} + \sigma_i(t)\underbrace{\sum_{k=N}^{i} \frac{f_k(t)\tau_k}{1 + f_k(t)\tau_k}\sigma_k(t)\varrho_{ik}}_{\text{non-zero for } i \geq N} \tag{3}$$

3 Futures convexity

The value of a future contract on a forward rate fixing at time t_i is given by the expectation of f_i in the spot measure (also known as the measure associated with the chosen numéraire being the continuously rolled up money market account)[2]:

$$\hat{f}_i = \mathsf{E}^*[f_i(t_i)] \tag{4}$$

In the following, we assume that in between two subsequent canonical fixing dates of adjacent forward Libor rates all overnight forward rates are also fixed and thus deterministic. This means that we are effectively using the measure associated with the selected numéraire being the *discretely* rolled up money market account. In practice, any model implementation's specific way of allowing for stub forward rates to continue to evolve stochastically beyond the canonical discrete forward rate's fixing date can be taken into account. A very simple estimation of the magnitude of the error thus incurred can be obtained by allowing the respective forward rate volatility functions $\sigma_i(t)$ to be non-zero not just until t_i, but until t_{i+1} in our integral formulæ presented in the following[3]. However, throughout all our tests, we found that for the purpose of futures convexity calculations the difference between expectations in the continuously compounded money market account measure and the discretely compounded money market account measure are negligible[4]

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[2]For a proof of this result, which was first published in [CIR81], see, for instance, theorem 3.7 in [KS98]

[3]Such a simplistic method to allow for the discrete spot stub rate $(1/P_1(t)-1)/(t_1-1)$ for $(t_1-t) < \tau_0$ to continue being stochastic until its residual term $(t_1 - t)$ finally vanishes is, strictly speaking, not arbitrage-free, but the violation is of such small magnitude that it does not constitute an enforceable arbitrage and thus is acceable for the purposes of the mentioned estimation.

[4]There were three kinds of tests we carried out to establish this result: first, we tested for the difference between three-monthly rolling and daily rolling in a one-factor Hull-White model. Then, we tested with a three-factor Hull-White model with significant differences in mean reversion level between the three factors. Thirdly, we tested for the difference with our own method of continuation of stochasticity of stub Libor rates beyond the fixing date of the canonical forward rate. Naturally, we also checked the differences as resulting from our analytical formulæ using the above suggested simplistic stub stochasticity continuation method of allowing $\sigma_i(t)$ to be non-zero until t_{i+1}.

In the discretely rolled up spot measure, we have

$$\frac{\mathrm{d}f_i(t)}{f_i(t)} = \mu_i^*(\boldsymbol{f}(t), t)\,\mathrm{d}t + \sigma_i(t)\,\mathrm{d}W_i^* \,, \qquad \text{with} \qquad \mu_i^*(\boldsymbol{f}(t), t) = \sigma_i(t) \sum_{k=1}^{i} \frac{f_k(t)\tau_k}{1 + f_k(t)\tau_k} \sigma_k(t)\varrho_{ik}(t) \,. \tag{5}$$

In this setting, the lowest order futures convexity correction for the i-th forward rate is given by

$$\mathsf{E}^*[f_i(t_i)] \approx f_i(0) \cdot \mathrm{e}^{\int_{t=0}^{t=t_i} \mu_i^*(\boldsymbol{f}(0), t)\,\mathrm{d}t} \,. \tag{6}$$

For the Libor market model, the above approximation usually fails quite dramatically due to the fact that the drift expression $\mu_i^*(\boldsymbol{f}, t)$ is itself stochastic. One approach to remedy the situation is to apply the technique of iterated substitutions (also known as Itô-Taylor expansion).

3.1   Convexity conundrums

For the extended Vasicek (Hull-White) model, an exact expression for the futures price can be derived from the stochastic differential equation for the short rate

$$\mathrm{d}r = \lambda\left(\theta(t) - r\right)\mathrm{d}t + \sigma_{\mathrm{HW}}\mathrm{d}W \,.$$

In order to compare with the results published in 1997 by Kirikos and Novak [KN97] for a single factor model with constant diffusion coefficient σ_{HW} and constant mean reversion parameter λ, we briefly recall a few facts specific to the extended Vasicek model:

- The inverse forward discount factor $P(t, T_s, T_e) := P(t, T_s)/P(t, T_e)$ associated with a forward loan from T_s to T_e is drift free in the T_e forward measure generated by choosing the numéraire to be the T_e-maturing zero coupon bond $N(t) := P(t, T_e)$.

- The instantaneous relative volatility of a forward discount factor associated with a forward loan from T_s to T_e at time t is $\sigma(t, T_s, T_e) = \frac{\sigma_{\mathrm{HW}}}{\lambda}\left(\mathrm{e}^{-\lambda(T_s-t)} - \mathrm{e}^{-\lambda(T_e-t)}\right)$.

- The change of drift required when switching from the T_e forward measure to the spot measure is given by $\mathrm{d}W^{T_e} = \mathrm{d}W^{\mathrm{spot}} + \sigma(t, t, T_e)\mathrm{d}t$.

- The futures convexity correction for the Libor rate f fixing at time T_f and spanning the accrual period from T_s to T_e is given by $(1 + \tau\hat{f}) = (1 + \tau f(0))\,\mathrm{e}^C$ with

$$C = \int_0^{T_f} \sigma(t, T_s, T_e)\sigma(t, t, T_e)\mathrm{d}t = \frac{\sigma_{\mathrm{HW}}^2}{2\lambda^3}\mathrm{e}^{-\lambda(T_s+2T_e)}\left(\mathrm{e}^{\lambda T_f} - 1\right)\left(2\mathrm{e}^{\lambda T_e} - \mathrm{e}^{\lambda T_f} - 1\right)\left(\mathrm{e}^{\lambda T_e} - \mathrm{e}^{\lambda T_s}\right) \tag{7}$$

  wherein $\hat{f}$ represents today's fair price of the futures contract on f, and $\tau = T_e - T_s$.

- For $\lambda = 0$ and $T_f = T_s = T_e - \tau$, formula (7) simplifies to

$$C = \frac{\sigma_{\mathrm{HW}}^2}{2}T(T + 2\tau)\tau \tag{8}$$

- For $\lambda = 0$ and $T_f = T_s = T_e - \tau$, the price of a caplet struck at K is given by

$$P(0, T + \tau) \cdot \mathrm{B}(1 + \tau f, 1 + \tau K, \sigma_{\mathrm{HW}}\tau, T) \,, \tag{9}$$

  where $\mathrm{B}(F, K, \hat{\sigma}, T)$ is Black's formula.

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4 Itô-Taylor expansions for Libor in arrears

Before we proceed to the much more difficult case of the actual futures convexity calculation, we will first illustrate the Itô-Taylor expansion method using the simpler example of a forward contract on a Libor in arrears[5]. The risk-neutral price of a Libor in arrears contract is the expected value of a forward rate $f_i(t_i)$ in the forward measure associated with the numéraire by the zero coupon bond maturing at t_i, that is, $P_i(0) \cdot \mathsf{E}_{t_i}[f_i(t_i)]$, and in the Libor market model:

$$\mathsf{E}_{t_i}[f_i(t_i)] = \mathsf{E}_{t_{i+1}}\left[\frac{1 + f_i(t_i)\tau_i}{1 + f_i(0)\tau_i} \cdot f_i(t_i)\right] = \frac{f_i(0) + f_i(0)^2\tau_i \cdot \mathrm{e}^{\int_{t=0}^{t=t_i}\sigma_i(t)^2\mathrm{d}t}}{1 + f_i(0)\tau_i} \ . \tag{10}$$

In the t_i forward measure the forward rate f_i follows

$$\mathrm{d}f_i(t) = \sigma_i(t)^2 \frac{f_i(t)^2\tau_i}{1 + f_i(t)\tau_i}\mathrm{d}t + \sigma_i(t)f_i(t)\,\mathrm{d}\widetilde{W}_i \ . \tag{11}$$

Using an Itô-Taylor expansion of the drift term in equation (11), the following n-th order approximation can be obtained:

$$\mathsf{E}_{t_i}[f_i(t_i)] \approx f_i(0) \cdot \left(1 + \varepsilon_{\mathrm{LIA}}^{(n)}\right) \ , \qquad \text{with} \qquad \varepsilon_{\mathrm{LIA}}^{(n)} = \sum_{k=1}^{n}\frac{a_k}{f_i(0) \cdot k!}\left(\int_{t=0}^{t=t_i}\sigma_i(t)^2\mathrm{d}t\right)^k \ , \tag{12}$$

whereby

$$a_1 = \frac{f_i(0)^2\tau_i}{1 + f_i(0)\tau_i} \ , \qquad \text{and for } k \geq 2, \qquad a_k = a_1\frac{\mathrm{d}a_{k-1}}{\mathrm{d}f_i(0)} + \frac{1}{2}f_i(0)^2\frac{\mathrm{d}^2 a_{k-1}}{\mathrm{d}f_i(0)^2} \ . \tag{13}$$

Based on our experience with approximate expansions, we also evaluated the following lognormal modification to (12):

$$\mathsf{E}_{t_i}[f_i(t_i)] \approx f_i(0) \cdot \mathrm{e}^{\varepsilon_{\mathrm{LIA}}^{(n)}} \ . \tag{14}$$

Heuristically, we find that this modification does indeed improve the accuracy of the approximation when the series is truncated early, which is, intuitively, in agreement with our understanding that the forward rate is still close to being lognormally distributed. In table 1, we show the accuracy of approximations (12) and (14). The test data are $t_i = 5$, $f_i(0) = 5\%$ years, $\tau_i = 0.25$ and $\sigma_i(t) = 40\% \ \forall t$. Us-

(1bp $\hat{=}$ 0.01%)	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Approximation (12)	-2.627bp	-0.651bp	-0.125bp	-0.019bp	-0.003bp
Approximation (14)	-2.602bp	-0.603bp	-0.069bp	0.038bp	0.055bp

Table 1: Absolute accuracy of approximations (12) and (14) for the fair strike of a 3 month Libor in arrears contract with $T = 5$ years until fixing, the forward Libor level being $f = 5\%$, and constant instantaneous lognormal volatility of $\sigma(t) = 40\%$.

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[5]Confusingly, *Libor in arrears* means that a London-interbank-offered-rate fixing dependent coupon is paid at the beginning of the associated accrual period, instead of the conventional payment at the end. Confusion sometimes arises from the fact that the term *arrears* refers to the *fixing* time, not the payment time. A Libor dependent coupon is usually computed as a (possibly nonlinear) function of the Libor rate that was fixed at the beginning of the associated accrual period. When the Libor rate fixed at the end of a coupon's accrual period determines the payoff, the contract usually has the attribute *in arrears*, hence the nomenclature *Libor in arrears*.

ing (10), the exact value is 5.07565%. The approximation (12) converges to the exact value, whereas (14) improves the accuracy up until 3rd order whereafter it converges to a slightly higher level in the limit of $n \to \infty$ (in fact still well under 0.1bp above the analytically exact result). This is a typical feature of asymptotic methods that are designed to be accurate only within a given expansion order, are allowed to diverge, but often have the advantage that at a low expansion order they are more accurate than other, convergent, expansions.

Given the near-lognormality of the Libor in arrears, an intuitively appealing alternative is to apply the Itô-Taylor expansion to $X_i(t) := \ln(f_i(t))$. Using the Itô formula, $X_i(t)$ follows the stochastic differential equation

$$\mathrm{d}X_i(t) = \sigma_i(t)^2 \left(\frac{\mathrm{e}^{X_i(t)}\tau_i}{1 + \mathrm{e}^{X_i(t)}\tau_i} - \frac{1}{2} \right) \mathrm{d}t + \sigma_i(t)\mathrm{d}\widetilde{W}_i . \tag{15}$$

$$\mathsf{E}_{t_i}[f_i(t_i)] \approx f_i(0) \cdot \mathrm{e}^{\varepsilon_{\ln(\mathrm{LIA})}} , \tag{16}$$

where

$$\varepsilon_{\ln(\mathrm{LIA})} := \frac{f_i(0)\tau_i}{1 + f_i(0)\tau_i} \int\limits_{t=0}^{t=t_i} \sigma_i(t)^2 \mathrm{d}t . \tag{17}$$

As we can see, $\varepsilon_{\ln(\mathrm{LIA})}$ does not contain any higher order terms. This is because the expression

$$\frac{\mathrm{e}^{X_i(t)}\tau_i}{1 + \mathrm{e}^{X_i(t)}\tau_i} \tag{18}$$

is drift-free when X is governed by the stochastic differential equation (15). Approximation (16) is in fact nothing but the first order of the infinite series (14), i.e.

$$\varepsilon_{\ln(\mathrm{LIA})} = \varepsilon_{\mathrm{LIA}}^{(1)} , \tag{19}$$

which explains the somewhat counterintuitive observation that the initial lognormal transformation of $f_i(t)$ is, alas, of little help.

5   Monte Carlo simulations to compute the futures price

By simulating the dynamics (5), we can find the futures price numerically in the spot measure. Otherwise, using the following relationship, it can be priced in the forward measure associated with the numéraire given by the zero coupon bond maturing at t_i.

$$\mathsf{E}^*[f_i(t_i)] = P_{i+1}(0) \cdot \mathsf{E}_{t_{i+1}} \left[f_i(t_i) \cdot \prod_{j=0}^{i} (1 + f_j(t_j)\tau_j) \right] . \tag{20}$$

It is by the aid of the right hand side of equation (20) that we computed our numerical reference values. For details of our simulation framework, see [Jäc02].

6   Matsumoto's formula

Matsumoto's approximation for the futures convexity[6] can be obtained by realising

$$P_{i+1}(0) \cdot \mathsf{E}_{t_{i+1}} \left[f_i(t_i) \cdot \prod_{j=0}^{i} (1 + f_j(t_j)\tau_j) \right] = f_i(0) + P_{i+1}(0) \cdot \mathsf{Cov}_{t_{i+1}} \left[f_i(t_i), \prod_{j=0}^{i} (1 + f_j(t_j)\tau_j) \right] , \tag{21}$$

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[6]equation (21) in [Mat01]

by expanding the product in the right hand side of (21) up to first order in forward rates, and by approximating each of the remaining forward rates as a lowest (non-trivial) order forward-measure-drift-adjusted lognormal variate. This gives us Matsumoto's formula

$$\mathsf{E}^*[f_i(t_i)] \approx f_i(0)\left(1 + \varepsilon^{(\text{Matsumoto})}\right) \tag{22}$$

where

$$\varepsilon^{(\text{Matsumoto})} := P_{i+1}(0) \cdot \sum_{j=1}^{i} f_j(0)\tau_j \cdot \mathrm{e}^{\int_{t=0}^{t=t_i} \mu_j^{(t_{i+1})}(\boldsymbol{f}(0),t)\,\mathrm{d}t} \left(\mathrm{e}^{\int_{t=0}^{t=t_i}\sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\,\mathrm{d}t} - 1\right). \tag{23}$$

with $\mu_j^{(t_{i+1})}(\boldsymbol{f}(0),t)$ defined as in equation (3). Please note that our convention of forward rate volatilities dropping to zero after the respective fixing time automatically extends to the drift functions since all drift terms are ultimately driven by covariance expressions.

7 Itô-Taylor expansion for futures

Unfortunately, unlike the Libor in arrears case, there is no closed form solution for the risk-neutral futures value (4) in the Libor market model framework. Similarly to the Libor in arrears case, though, an attempt to approximate the fair value of the futures contract by means of an Itô-Taylor expansion of the drift of the logarithm of the forward rate leads to rather poor approximations. Just as in the previous section, we therefore base our expansion on the *absolute drift term* of equation (5), i.e. $f_i(t) \cdot \mu_i^*(\boldsymbol{f},t)$. Applying the Itô-Taylor expansion method[7], the following n-th order approximation is derived in appendix A:

$$\mathsf{E}^*[f_i(t_i)] \approx f_i(0)\left(1 + \varepsilon^{(n)}\right) \tag{24}$$

where

$$\varepsilon^{(n)} := \sum_{k=1}^{n}\sum_{j=1}^{i}\frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^k}\frac{1}{k!}\left(\int_{t=0}^{t=t_i}\sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\,\mathrm{d}t\right)^k + \frac{3}{2}\sum_{k=2}^{n}\frac{1}{k!}\left(\sum_{j=1}^{i}\frac{f_j(0)\tau_j}{1+f_j(0)\tau_j}\int_{t=0}^{t=t_i}\sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\,\mathrm{d}t\right)^k. \tag{25}$$

Since the forward rate in the spot measure is still somewhat close to lognormal, we propose again the following lognormal modification to (24) when n is small:

$$\mathsf{E}^*[f_i(t_i)] \approx f_i(0) \cdot \mathrm{e}^{\varepsilon^{(n)}}, \tag{26}$$

In table 2, we compare the accuracy of the approximate formulæ (24) and (26). The test data are $t_i = 5$

(1bp $\hat{=}$ 0.01%)	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Approximation (24)	-28.3bp	-9.9bp	-6.9bp	-6.4bp	-6.4bp
Approximation (26)	-25.5bp	-4.6bp	-1.1bp	-0.6bp	-0.5bp

Table 2: Absolute accuracy of approximations (24) and (26) for the fair value of a futures contract on a 3 month Libor rate with $T = 5$ years until fixing, the forward Libor level being $f = 5\%$, constant instantaneous lognormal volatility of $\sigma(t) = 40\%$ for all forward rates, and perfect instantaneous correlation, i.e. $\varrho_{ij}(t) = 1 \,\forall\, i, j, t$.

years with three month canonical periods, $f_i(0) = 5\% \,\forall\, i$, $\sigma_i(t) = 40\% \,\forall\, i, t$ and $\varrho_{ij}(t) = 1 \,\forall\, i, j, t$. The Monte Carlo value is obtained as 5.810%. We find that formula (26) gives a good approximation.

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[7]see [KP99] for details on Itô-Taylor expansions and [Kaw02, Kaw03] for its application to the Libor market model.

8 Displaced diffusion extension

In this section, we consider the Libor market model with a displaced diffusion. Similar to the framework discussed in [Jäc03], we assume that in the spot measure the forward rate follows

$$\frac{d(f_i + s_i)}{f_i + s_i} = \mu_i(\boldsymbol{f}, \boldsymbol{s}, t)\, dt + \sigma_i(t)\, dW_i^* \,, \qquad \text{where} \qquad \mu_i^*(\boldsymbol{f}(t), t) = \sigma_i(t) \sum_{k=1}^{i} \frac{(f_k(t) + s_k)\tau_k}{1 + f_k(t)\tau_k} \sigma_k(t)\varrho_{ik} \,, \tag{27}$$

and $s_i := -|f_i| \log_2 Q_i$ with $Q_i \in (0, 2)$. The flexibility of this framework allows us to change the forward rate dynamics gradually from the pure lognormal model over to the behaviour under the Hull-White model by decreasing Q from 1 to a number very close to zero (the precise number to match the Hull-White model depends on the level of interest rates and volatilities). Adapting our previous result to the stochastic differential equation (27) we obtain another n-th order approximation:

$$\mathsf{E}^*[f_i(t_i)] \approx (f_i(0) + s_i)\left(1 + \varepsilon_{\mathrm{DD}}^{(n)}\right) - s_i \,. \tag{28}$$

where

$$\varepsilon_{\mathrm{DD}}^{(n)} := \sum_{k=1}^{n} \sum_{j=1}^{i} \frac{(f_j(0)+s_j)\tau_j}{(1+f_j(0)\tau_j)^k} \frac{1}{k!} \left(\int_{t=0}^{t=t_i} \sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\, dt\right)^k + \frac{3}{2} \sum_{k=2}^{n} \frac{1}{k!} \left(\sum_{j=1}^{i} \frac{(f_j(0)+s_j)\tau_j}{1+f_j(0)\tau_j} \int_{t=0}^{t=t_i} \sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\, dt\right)^k \,. \tag{29}$$

Again, we also consider a lognormal modification to (28):

$$\mathsf{E}^*[f_i(t_i)] \approx (f_i(0) + s_i) \cdot e^{\varepsilon_{\mathrm{DD}}^{(n)}} - s_i \,. \tag{30}$$

We show in table 3 the accuracy of approximate formulæ (28) and (30) in comparison. The test data

(1bp $\hat{=}$ 0.01%)	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Approximation (28)	-5.8bp	-0.5bp	-0.4bp	-0.3bp	-0.3bp
Approximation (30)	-4.5bp	1.1bp	1.3bp	1.3bp	1.3bp
Approximation (31)	-5.1bp	0.2bp	0.4bp	0.5bp	0.5bp

Table 3: Absolute accuracy of approximations (28), (30), and (31) for the fair value of a futures contract on a 3 month Libor rate with $T = 5$ years until fixing, the forward Libor level being $f = 5\%$, constant instantaneous lognormal volatility of $\sigma(t) = 40\%$ for all forward rates, $Q = 1/2$, and perfect instantaneous correlation, i.e. $\varrho_{ij}(t) = 1 \ \forall\, i, j, t$.

are $Q = 1/2$, $t_i = 5$ years with three month canonical periods, $f_i(0) = 5\% \ \forall\, i$, $\sigma_i(t) = 40\% \ \forall\, i, t$ and $\varrho_{ij}(t) = 1 \ \forall\, i, j, t$. Using (20), the Monte Carlo value is obtained as 5.566%. We find that formula (28) seems to give a good approximation. One reason that the lognormal modification does not work perfectly is that, when $Q = 1/2$, the distribution of the forward rate is closer to normal than when $Q = 1$. To take this effect into account, we propose the following modification to formula (30):

$$\mathsf{E}^*[f_i(t_i)] \approx (f_i(0) + s_i)\left((1 - \log_2 Q) \cdot e^{\frac{\varepsilon_{\mathrm{DD}}^{(n)}}{1 - \log_2 Q}} + \log_2 Q\right) - s_i \,. \tag{31}$$

The accuracy for approximation (31) is also given in table 3. We find that it is, overall, an improvement on (30).

9   Numerical results

In order to demonstrate the accuracy of our expansions, we had to make a choice of relevant scenarios. This choice is made particularly hard by the fact that the intrinsic flexibility of the Libor market model allows for extremely wide ranges of volatility and correlation configurations. We therefore decided to choose nine sample sets that cover to some extent almost all of the market-observable features and commonly used modelling paradigms:-

- $Q = 1$: Lognormal discrete forward rate distributions. This setting has similarities with the Black-Karasinski model [BK91], albeit on a discrete basis, and is, of course, the configuration of the conventional Libor market Model.

- $Q = 1/2$: The implied smile of caplets is extremely similar to that resulting from a Cox-Ingersoll-Ross model [CIR85].

- $Q = 10^{-8}$: Discrete forward rates in the Hull-White model have a distribution that is consistent with the displaced diffusion model for a very small Q coefficient (which depends on the level of interest rates). This is easy to understand if we recall that zero-coupon bonds are lognormally distributed in the Hull-White model and thus the inverse of a discount factor over any Libor period minus the constant 1, which amounts to the respective Libor rate times its accrual factor, is a shifted lognormal variate.

Orthogonally to these three smile/skew approximations corresponding to the three modelling concepts of lognormal, square root, and normal distributions, we chose the three market scenarios of:-

- Medium level interest rates at 5%, slightly elevated interest rate volatilities at 40%, and perfect yield curve correlation corresponding to the results from a one-factor model analysis. This scenario is somewhat similar to the current interest rate markets in GBP and EUR, albeit that we have raised volatilities a little to emphasise the observable effects.

- Low interest rates at 1%, elevated volatilities around 60%, and perfect correlation. This scenario is reminiscent of the current USD environment for short maturities, only that our volatilities are slightly lower than observed in that market.

- Medium rates at 5%, volatilities around 40%, and strong, perhaps even slightly exaggerated, de-correlation. This scenario is similar to the current long-dated futures market in USD.

These three modelling setups and market scenarios form a matrix of nine experiments whose results we report.

In figures 1 to 6, we show the numerical and analytical results in comparison for a number of different scenarios and times to expiry t_i with perfect correlation among all the quarterly forward rates (i.e. $\tau_i = 1/4$ and $\varrho_{ij} = 1$). In order to make the results for different values of Q compatible, we hereby always kept the price of an at-the-money caplet (expressed as its Black volatility $\hat{\sigma}_i$) constant by adjusting the displaced diffusion coefficients throughout the figures 1 to 3, and 4 to 6, respectively.

Please note that the apparent fluctuations in the results aren't actually due to residual numerical noise of the Monte Carlo results as one might at first suspect. Instead, they are caused by the fact that the presented data were computed taking into account the differences in the quarterly periods as they occur in the financial markets. Since the numerical and analytical results are overall respectively very close indeed, the small differences in the number of days in each 3-month accrual period and volatility interval account for the noticeable discrepancies. This effect is probably most readily visible in figure 3 where we have a pronounced annual periodicity of the peaks in the numerical differences.
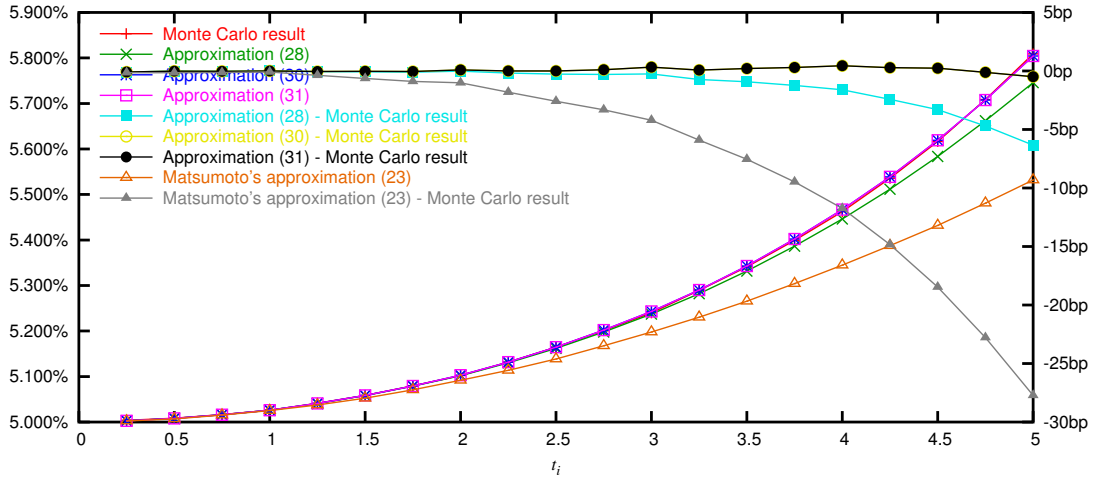
Figure 1: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, and $Q = 1$. The data for (30) and (31) are superimposed because they are identical for $Q = 1$.
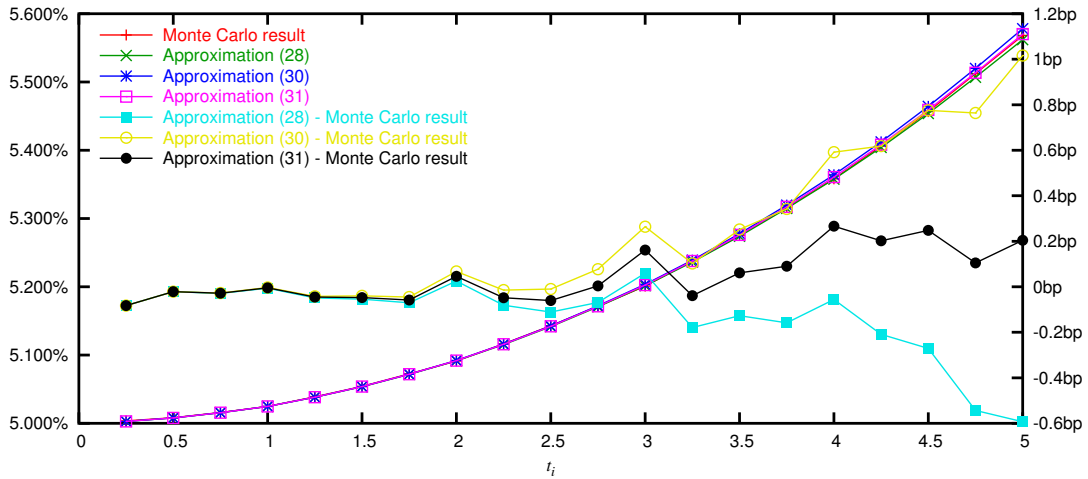

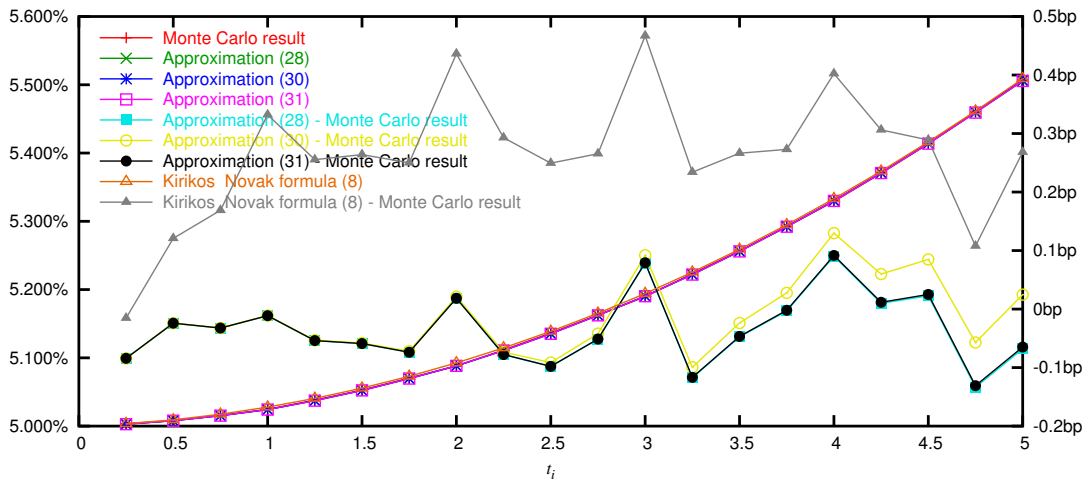Figure 2: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, and $Q = 1/2$.

Figure 3: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, and $Q = 10^{-8}$. The Hull-White coefficient used for formula (8) was $\sigma_{\mathrm{HW}} = 1.91\%$.

We should add, that, albeit that we didn't show the respective figures, for $Q = 1$, when direct comparison of our approximations and Matsumoto's formula (22) is possible, our first order expansions give results very similar to Matsumoto's result. As the reader can see, for maturities beyond a couple of years, Matsumoto's formula starts to deviate from tradeable accuracy, and it is this discrepancy beyond two years that ultimately prompted us to derive higher order approximations.

Since the price of a canonical caplet in the Libor market model for constant instantaneous volatility is given by

$$P(0, t_{i+1}) \cdot \mathbf{B}(f_i, K, \hat{\sigma}_i, t_i) \,, \tag{32}$$

we can use equation (9) to impute what constant σ_{HW} we would have to use in a single factor Hull-White model with zero mean reversion in order to ensure that both models match the same at-the-money caplet prices. Using this method for at-the-money caplet calibration, we then used formula (8) to add the convexity correction line as it would result from a single factor Hull-White model to the scenarios shown in figures 3 and 6 since the case $Q = 10^{-8}$ is virtually equivalent to normally distributed Libor rates. It is noteworthy that a small difference can only be observed in the example where forward rates are around 1%. For lower interest rates with even higher volatilities, the distinction becomes even more pronounced which is of particular importance in the current JPY interest rate markets.
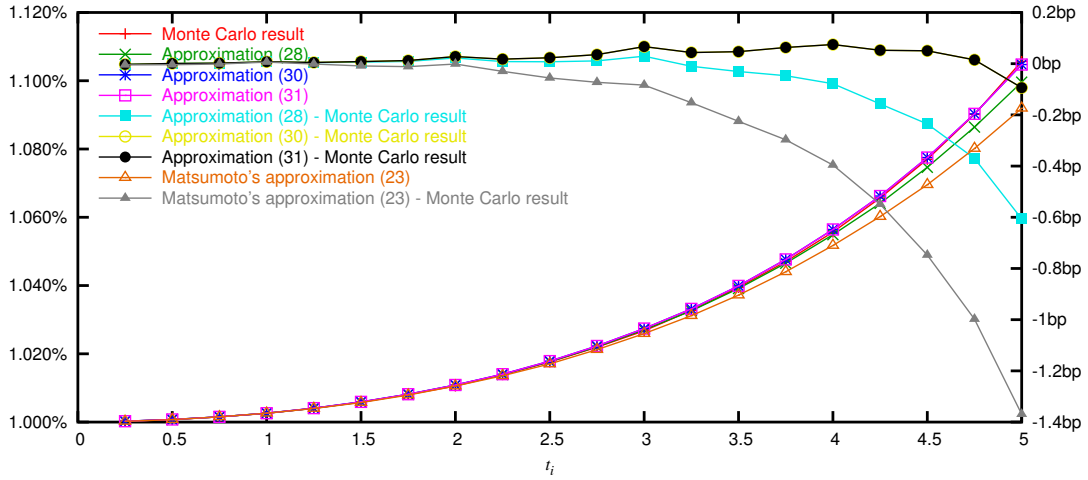

Figure 4: Numerical and analytical results for $f_i = 1\%$, $\hat{\sigma}_i = 60\%$, and $Q = 1$. The data for (30) and (31) are superimposed because they are identical for $Q = 1$.
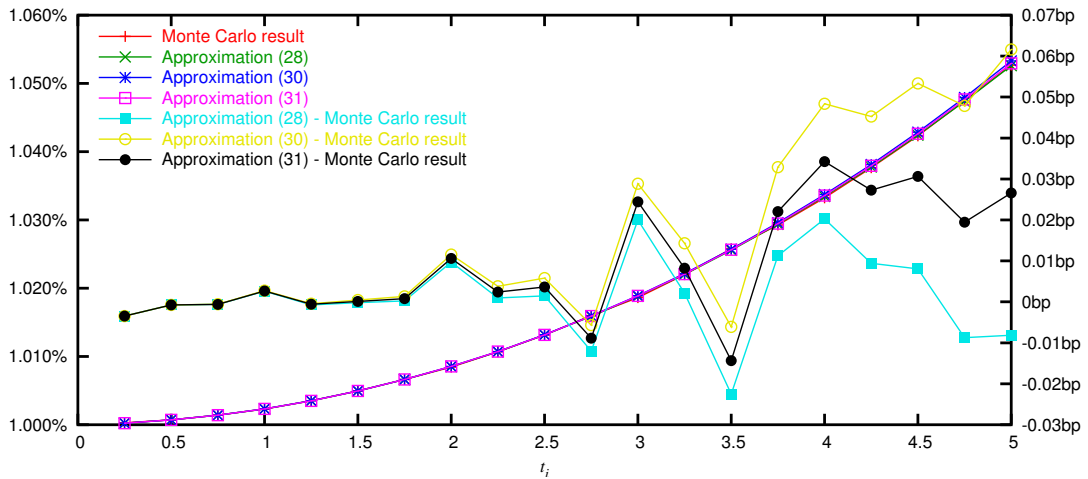

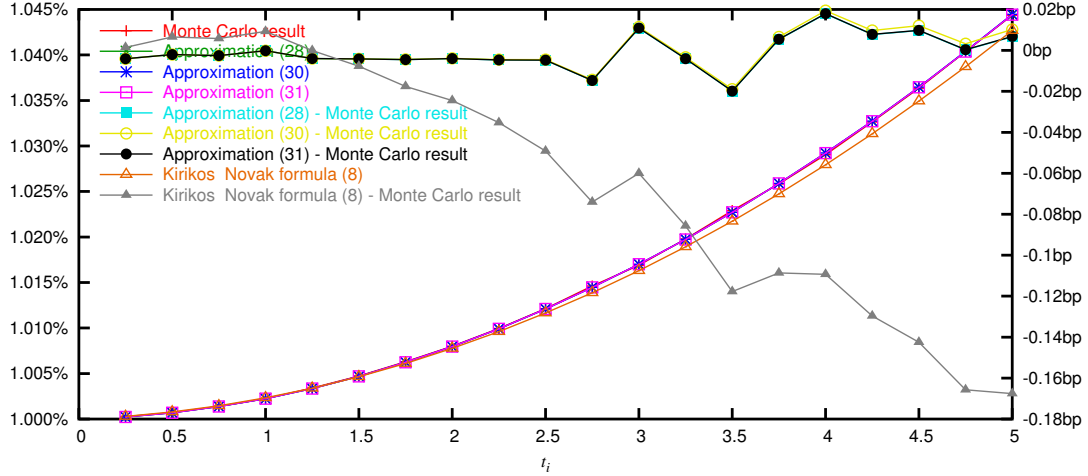Figure 5: Numerical and analytical results for $f_i = 1\%$, $\hat{\sigma}_i = 60\%$, and $Q = 1/2$.

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Figure 6: Numerical and analytical results for $f_i = 1\%$, $\hat{\sigma}_i = 60\%$, and $Q = 10^{-8}$. The Hull-White coefficient used for formula (8) was $\sigma_{\mathrm{HW}} = 0.56\%$.

At this point, we owe the reader a demonstration that the presented convexity formulæ do not only work in the case of perfect correlation. For this purpose, we use the following time-homogenous and time-constant correlation function:

$$\rho_{ij} = \mathrm{e}^{-\beta|t_i - t_j|} \tag{33}$$

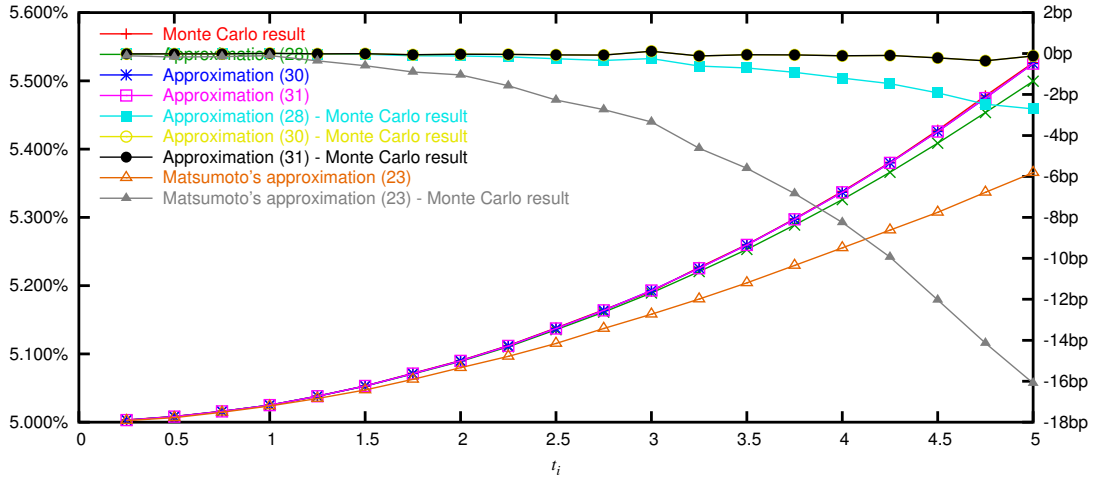In order to slightly exaggerate the effect of decorrelation, we set the parameter $\beta = 1/4$. This means that

Figure 7: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, $\beta = 1/4$, and $Q = 1$. The data for (30) and (31) are superimposed because they are identical for $Q = 1$.

forward rates whose fixings are approximately 5 years apart, appear to have a correlation of only around 30% which is somewhat smaller than what we would estimate from time series information in the major interest rate currencies. We notice in figures 7 to 9 that, overall, decorrelation poses no extra difficulty for the accuracy of the approximation. Given our past experience with swaption [JR00, Kaw02, Kaw03] and non-canonical caplet [Jäc03] expansions, however, this result is not surprising at all.

10   Conclusion

We have presented new formulæ for futures convexity derived within the framework of a Libor market model allowing for a skew in implied volatilities consistent with displaced diffusion equations. The
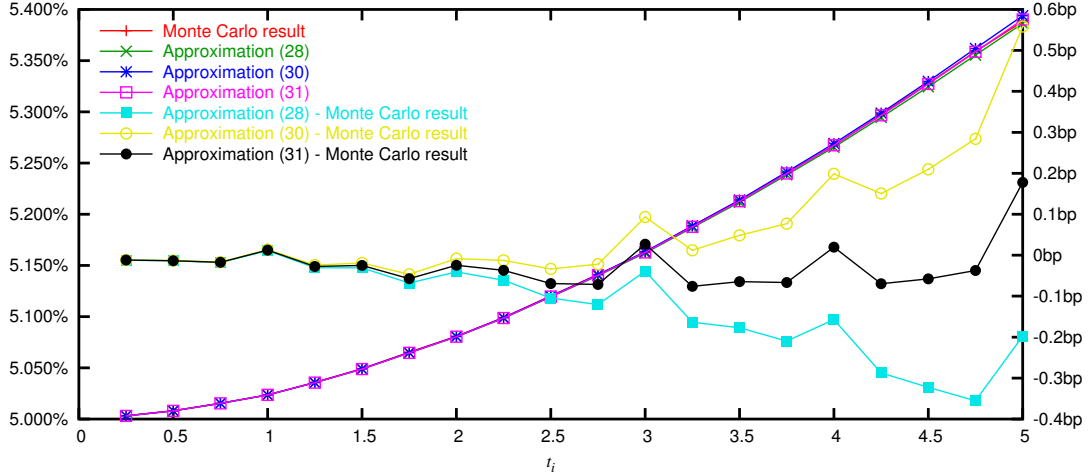
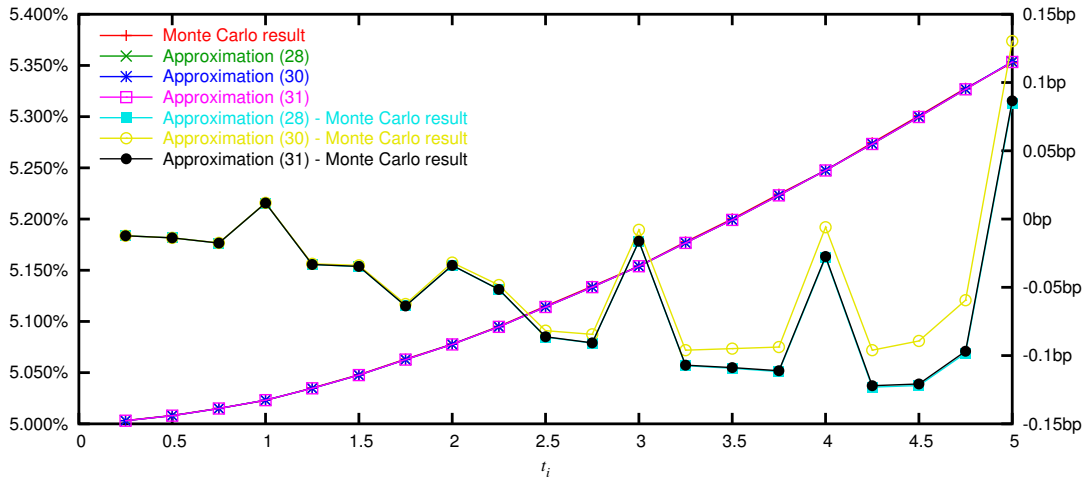Figure 8: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, $\beta = 1/4$, and $Q = 1/2$.

Figure 9: Numerical and analytical results for $f_i = 5\%$, $\hat{\sigma}_i = 40\%$, $\beta = 1/4$, and $Q = 10^{-8}$.

approximations were computed with a combination of Itô-Taylor expansions, heuristic truncations and structural modifications. The results were tested against a variety of market scenarios and were found to be highly accurate and reliable, and can thus be used as part of a yield curve stripping algorithm. What's more, the developed techniques are applicable to other problems such as the handling of quanto effects and the approximation of plain vanilla FX option prices in a multi-currency Libor market model, which we will demonstrate in a forthcoming article.

Finally, we should mention that we also tested the presented futures price approximations for many real market calibration scenarios with contemporary yield curves and volatility surfaces, and that we always found the modified displaced diffusion approximation (31) to work extremely well. In fact, we chose the presented artificial test cases for presentation since they comprise significantly more difficult scenarios than real market scenarios. All our findings are in agreement with our usual observation that decorrelation helps any assumptions akin to averaging effects on which some of the simplifications used in our expansions explained in the appendix are based. In summary, we find that equation (31) for a displaced-diffusion Libor market model is an extremely robust and highly accurate formula for all major interest markets and correlation assumptions.

A Derivation of approximation (24)

The starting point of the approximation is the following general principle. Given a process $\boldsymbol{x}$ governed by the stochastic differential equation

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{\nu}(\boldsymbol{x},t)\mathrm{d}t + A\,\mathrm{d}\boldsymbol{W} \tag{34}$$

whereby $A = A(t)$ represents the pseudo-square root of the instantaneous covariance matrix[8], i.e.

$$A(t) \cdot A^\top(t) \;=\; C(t)\,, \tag{35}$$

with $c_{ij}(t) = \sigma_i(t)\rho_{ij}(t)\sigma_j(t)$, we have under some suitable regularity conditions conditions[9]

$$\mathsf{E}^*[\boldsymbol{x}(t)] \;=\; \boldsymbol{x}(0) + \mathsf{E}^*\left[\int_{u=0}^t \mathrm{d}\boldsymbol{x}(u)\right] \;=\; \boldsymbol{x}(0) + \int_{u=0}^t \mathsf{E}^*[\mathrm{d}\boldsymbol{x}(u)] \;=\; \boldsymbol{x}(0) + \int_{u=0}^t \mathsf{E}^*[\boldsymbol{\nu}(\boldsymbol{x}(u),u)]\,\mathrm{d}u\,. \tag{36}$$

For any process $\boldsymbol{y}(\boldsymbol{x},t)$ satisfying certain benevolence conditions (in particular that $\boldsymbol{y}$ is finite, integrable, and at least piecewise differentiable in $\boldsymbol{x}$), we can apply Itô's lemma to obtain

$$\begin{aligned}
\mathsf{E}^*[\boldsymbol{y}(\boldsymbol{x}(t),t)] \;&=\; \boldsymbol{y}(\boldsymbol{x}(0),0) + \mathsf{E}^*\left[\int_0^t \partial_u \boldsymbol{y}(\boldsymbol{x}(u),u)\,\mathrm{d}u\right] + \mathsf{E}^*\left[\int_0^t \mathrm{D}_{\boldsymbol{\nu},C}(u)\cdot\boldsymbol{y}(\boldsymbol{x}(u),u)\,\mathrm{d}u\right] \\[2mm]
&=\; \boldsymbol{y}(\boldsymbol{x}(0),0) + \int_0^t \mathsf{E}^*[\partial_u \boldsymbol{y}(\boldsymbol{x}(u),u)]\mathrm{d}u + \int_0^t \mathsf{E}^*[\mathrm{D}_{\boldsymbol{\nu},C}(u)\cdot\boldsymbol{y}(\boldsymbol{x}(u),u)]\,\mathrm{d}u
\end{aligned} \tag{37}$$

where we have defined the drift operator[10]

$$\mathrm{D}_{\boldsymbol{\nu},C}(t) \;=\; \left(\boldsymbol{\nu}^\top(\boldsymbol{x},t) + \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{x}}^\top\cdot C(t)\right)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\,. \tag{38}$$

Note that in the specific cases of (5) and (27), all explicit dependence of the absolute drift on t is through $C(t)$ since the drift is an explicit function of the state variables and covariance terms, i.e. the sole direct dependence of $\boldsymbol{\nu}$ on t is because it contains terms involving $C(t)$. In other words, for (27) and (5), we have $\partial_t C(t) = 0 \rightarrow \partial_t \boldsymbol{\nu} = 0$.

In the following, all expectations that we need to compute are for functions that decompose into a sum over separable terms, i.e. for processes of the form

$$y_i(\boldsymbol{x},t) \;=\; \sum_j \xi_{ij}(\boldsymbol{x})\theta_{ij}(t) \tag{39}$$

for some arbitrary functions $\xi_{ij}(\boldsymbol{x})$ and $\theta_{ij}(t)$. For such processes, equation (37) simplifies:

$$\begin{aligned}
\mathsf{E}^*[y_i(\boldsymbol{x}(t),t)] \;&=\; \sum_j \theta_{ij}(t)\mathsf{E}^*[\xi_{ij}(\boldsymbol{x}(t))] \\[2mm]
&=\; \sum_j \theta_{ij}(t)\cdot\left(\xi_{ij}(\boldsymbol{x}(0)) + \int_0^t \mathsf{E}^*[\mathrm{D}_{\boldsymbol{\nu},C}(u)\cdot\xi_{ij}(\boldsymbol{x}(u))]\,\mathrm{d}u\right) \\[2mm]
&=\; y_i(\boldsymbol{x}(0),t) + \int_0^t \mathsf{E}^*[\mathrm{D}_{\boldsymbol{\nu},C}(u)\cdot y_i(\boldsymbol{x}(u),t)]\,\mathrm{d}u
\end{aligned} \tag{40}$$

Thus,

$$\mathsf{E}^*[\boldsymbol{y}(\boldsymbol{x}(t),t)] \;=\; \boldsymbol{y}(\boldsymbol{x}(0),t) + \int_0^t \mathsf{E}^*[\mathrm{D}_{\boldsymbol{\nu},C}(u)\cdot\boldsymbol{y}(\boldsymbol{x}(u),t)]\,\mathrm{d}u\,. \tag{41}$$

---

[8]The matrix $A(t)$ may also be referred to as the *dispersion matrix*.

[9]See [KP99] or [KS91] for technical details on the applicability of Itô-Taylor expansions.

[10]For the sake of brevity, we only mention the time variable as an explicit dependency of the drift operator $\mathrm{D}_{\boldsymbol{\nu},C}(t)$.

Since $D_{\boldsymbol{\nu},C}(u) \cdot \boldsymbol{x} = \boldsymbol{\nu}(\boldsymbol{x}(u), u)$, we now obtain a rule for iterated substitutions

$$
\begin{align}
\mathsf{E}^*[\boldsymbol{x}(t)] &= \boldsymbol{x}(0) + \int_0^t \mathsf{E}^*[\boldsymbol{\nu}(\boldsymbol{x}(u_1), u_1)] \, \mathrm{d}u_1 \tag{42} \\
&= \boldsymbol{x}(0) + \int_0^t \boldsymbol{\nu}(\boldsymbol{x}(0), u_1) \, \mathrm{d}u_1 + \int_0^t \int_0^{u_1} \mathsf{E}^*[D_{\boldsymbol{\nu},C}(u_2) \cdot \boldsymbol{\nu}(\boldsymbol{x}(u_2), u_1)] \, \mathrm{d}u_2 \mathrm{d}u_1 \tag{43} \\
&= \boldsymbol{x}(0) + \int_0^t \boldsymbol{\nu}(\boldsymbol{x}(0), u_1) \, \mathrm{d}u_1 + \int_0^t \int_0^{u_1} D_{\boldsymbol{\nu},C}(u_2) \cdot \boldsymbol{\nu}(\boldsymbol{x}(0), u_1) \, \mathrm{d}u_2 \mathrm{d}u_1 \tag{44} \\
&\quad + \int_0^t \int_0^{u_1} \int_0^{u_2} \mathsf{E}^*\left[D_{\boldsymbol{\nu},C}^2(u_3) \cdot \boldsymbol{\nu}(\boldsymbol{x}(u_3), u_1) \right] \, \mathrm{d}u_3 \mathrm{d}u_2 \mathrm{d}u_1 \, ,
\end{align}
$$

which ultimately leads to the following *Itô-Taylor expansion for the expectation of $\boldsymbol{x}(t)$*:

$$
\mathsf{E}^*[\boldsymbol{x}(t)] = \boldsymbol{x}(0) + \sum_{k=1}^{\infty} \int_0^t \int_0^{u_1} \cdots \int_0^{u_{k-1}} D_{\boldsymbol{\nu},C}^{k-1}(u_k) \cdot \boldsymbol{\nu}(\boldsymbol{x}(0), u_1) \, \mathrm{d}u_k \cdots \mathrm{d}u_1 \, . \tag{45}
$$

For further details on Itô-Taylor expansions and their recursive definitions, see, for instance, chapter 5 in [KP99].

We now apply the above expansion technique to the absolute drift term of equation (5), that is,

$$
\nu_i(t) = f_i(t) \mu_i^*(\boldsymbol{f}, t) = f_i(t) \sum_{j=1}^i \frac{f_j(t) \tau_j}{1 + f_j(t) \tau_j} c_{ij}(t) \, . \tag{46}
$$

Only focussing on the explicit dependence of the drift term on the state variables, we compute

$$
D_{\boldsymbol{\nu},C} \cdot \left(\sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \right) = \sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \cdot \left[\frac{c_{ij}}{1 + f_j \tau_j} + \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l \tau_l} c_{il} + \sum_{l=1}^j \frac{f_l \tau_l}{1 + f_l \tau_l} \frac{c_{jl}}{1 + f_j \tau_j} - \frac{f_j \tau_j c_{jj}}{(1 + f_j \tau_j)^2} \right] \tag{47}
$$

where we have suppressed the explicit mentioning of dependencies on t. Assuming that $0 < f_l \tau_l \ll 1 \forall l$, the terms on the right hand side of (47) are sorted in descending order of magnitude. The second and third term within the brackets are of structural similarity whence we introduce the approximate simplification

$$
\begin{align}
\sum_{j=1}^i \frac{f_j \tau_j}{(1 + f_j \tau_j)^2} \sum_{l=1}^j \frac{f_l \tau_l}{1 + f_l \tau_l} c_{jl} &\approx \sum_{j=1}^i \frac{f_j \tau_j}{1 + f_j \tau_j} \sum_{l=1}^j \frac{f_l \tau_l}{1 + f_l \tau_l} c_{il} + \mathcal{O}\left((f\tau)^2 \cdot \max_{i,j>l} |c_{il} - c_{jl}| \right) + \mathcal{O}\left((f\tau)^3 \right) \tag{48} \\
&\approx \frac{1}{2} \sum_{j=1}^i \frac{f_j \tau_j}{1 + f_j \tau_j} \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l \tau_l} c_{il} \, . \tag{49}
\end{align}
$$

For positive rates and correlations, both steps (48) and (49) hereby lead to a small downwards bias. Dropping the least significant, i.e. the fourth, term in (47) leaves us with

$$
D_{\boldsymbol{\nu},C} \cdot \left(\sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \right) \approx \sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \cdot \left[\frac{c_{ij}}{1 + f_j \tau_j} + \frac{3}{2} \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l \tau_l} c_{il} \right] + \cdots \, . \tag{50}
$$

Having arrived at this level of approximate simplification, we now continue with the iterative substitution. Applying Itô's formula to the first term on the right hand side of equation (50) gives us

$$
\begin{align}
D_{\boldsymbol{\nu},C} \cdot \left(\sum_{j=1}^i \frac{f_i f_j \tau_j}{(1 + f_j \tau_j)^2} \right) &= \sum_{j=1}^i \frac{f_i f_j \tau_j}{(1 + f_j \tau_j)^2} \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l \tau_l} c_{il} + f_i \sum_{j=1}^i \frac{f_j \tau_j - f_j^2 \tau_j^2}{(1 + f_j \tau_j)^3} \sum_{l=1}^j \frac{f_l \tau_l}{1 + f_l \tau_l} c_{jl} \tag{51} \\
&\quad - f_i \sum_{j=1}^i \frac{2 f_j^2 \tau_j^2 - f_j^3 \tau_j^3}{(1 + f_j \tau_j)^4} c_{jj} + f_i \sum_{j=1}^i \frac{f_j \tau_j - f_j^2 \tau_j^2}{(1 + f_j \tau_j)^3} c_{ij} + \cdots \\
&\approx \sum_{j=1}^i \frac{f_i f_j \tau_j}{(1 + f_j \tau_j)^3} c_{ij} + \mathcal{O}\left((f\tau)^2 \right) + \cdots \, . \tag{52}
\end{align}
$$

Similarly, for the drift of the second term on the right hand side of equation (50) we obtain

$$
D_{\boldsymbol{\nu},C} \cdot \left(\sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l \tau_l} \right) \approx \sum_{j=1}^i \frac{f_i f_j \tau_j}{1 + f_j \tau_j} \sum_{l=1}^i \frac{f_l \tau_l}{1 + f_l(t) \tau_l} \sum_{m=1}^i \frac{f_m \tau_m}{1 + f_m \tau_m} c_{im} + \cdots \, . \tag{53}
$$

Using all of the above, and continuing the approximate iteration, our formula for the fair strike of the futures contract becomes

$$\mathsf{E}^*[f_i(t_i)] \quad \approx \quad f_i(0) \cdot (1 + \varepsilon) \tag{54}$$

with

$$
\begin{aligned}
\varepsilon \;=\; & \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} \int_0^{t_i} c_{ij}(u_1)\,du_1 \\
& + \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^2} \int_0^{t_i} \left(\int_0^{u_1} c_{ij}(u_2)\mathrm{d}u_2 \right) c_{ij}(u_1)\mathrm{d}u_1 \\
& + \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^3} \int_0^{t_i} \left(\int_0^{u_1} \left(\int_0^{u_2} c_{ij}(u_3)\mathrm{d}u_3 \right) c_{ij}(u_2)\mathrm{d}u_2 \right) c_{ij}(u_1)\mathrm{d}u_1 \\
& + \cdots \\
& + \frac{3}{2} \int_0^{t_i} \left(\int_0^{u_1} \sum_{l=1}^{i} \frac{f_l(0)\tau_l}{1+f_l(0)\tau_l} c_{il}(u_2)\mathrm{d}u_2 \right) \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} c_{ij}(u_1)\mathrm{d}u_1 \\
& + \frac{3}{2} \int_0^{t_i} \left(\int_0^{u_1} \left(\int_0^{u_2} \sum_{m=1}^{i} \frac{f_m(0)\tau_m}{1+f_m(0)\tau_m} c_{im}(u_3)\mathrm{d}u_3 \right) \sum_{l=1}^{i} \frac{f_l(0)\tau_l}{1+f_l(0)\tau_l} c_{il}(u_2)\mathrm{d}u_2 \right) \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} c_{ij}(u_1)\mathrm{d}u_1 \\
& + \cdots \\
\;=\; & \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} \int_0^{t_i} c_{ij}(u_1)\,du_1 + \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^2} \frac{1}{2} \left(\int_0^{t_i} c_{ij}(u_1)\mathrm{d}u_1 \right)^2 + \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^3} \frac{1}{6} \left(\int_0^{t_i} c_{ij}(u_1)\mathrm{d}u_1 \right)^3 + \cdots \\
& + \frac{3}{2} \left[\frac{1}{2} \left(\sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} \int_0^{t_i} c_{ij}(u_1)\mathrm{d}u_1 \right)^2 + \frac{1}{6} \left(\sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} \int_0^{t_i} c_{ij}(u_1)\mathrm{d}u_1 \right)^3 + \cdots \right] \\
\;=\; & \sum_{k=1}^{\infty} \sum_{j=1}^{i} \frac{f_j(0)\tau_j}{(1+f_j(0)\tau_j)^k} \frac{1}{k!} \left(\int_0^{t_i} \sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\,\mathrm{d}t \right)^k + \frac{3}{2} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{i} \frac{f_j(0)\tau_j}{1+f_j(0)\tau_j} \int_0^{t_i} \sigma_i(t)\sigma_j(t)\varrho_{ij}(t)\,\mathrm{d}t \right)^k .
\end{aligned}
\tag{55}
$$

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