Splitting the core

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The introduction of quoted prices for tranches of credit default obligations such as the Dow Jones iTraxx created the need for calibration of all CDO models to market observable figures. Whilst previously the market participants were used to estimating correlation figures and then simply computing the value of a new structure using their preferred analytical or numerical procedure, there is now a need to fit any remaining free parameters in the model to match the value of relevant hedge instruments. The most common approach for the pricing of basket credit derivatives is probably the connection of individual default time distributions with a copula based on one or more common factors. In this case, the calculation of the loss distribution can be decomposed into a sequence of orthogonal quadratures [Vas87, Li00, HW03]. Specifically, at the heart of the procedure is a variate y_i that is composed of the sum of independent factors:

$$y_i = \sum_{k=1}^{n_{\mathrm{f}}} x_k a_{ik} + \varepsilon_i \; . \tag{1}$$

If we choose the distributions of the systemic factor variables x_k and the idiosyncratic risk factor ε_i such that we know the law of y_i, the calculation of the time-discretised loss distribution can be implemented semi-analytically. When we make the simplifying assumption that all correlation coefficients are the same, say ϱ, and that there is only one systemic factor, all the coefficients a_i must be equal to $\sqrt{\varrho}$, thus resolving any ambiguity. This method has now become commonplace in the quotation of correlation numbers for baskets whose composition and whose constituents' individual credit default swap rates are quoted on indices. As things go in the derivatives business, the market has evolved further and it has been found that a different correlation figure ϱ is needed for any individual tranche of a CDO [McG04]. Several mechanisms have been suggested to match the market-observed *base correlation* skew, but in this note we want to focus on the concept of *correlation dispersion*, i.e. we want to be able to use a correlation matrix with a realistic interdepence structure as opposed to the flat correlation assumption in the straight-forward base correlation methodology. Gregory and Laurent [GL04] consider a structure built from groups specifying intra- and inter-group correlation coefficients. Perhaps more generally,

Andersen, Sidenius, and Basu [ASB03], use an iterative procedure that requires an eigensystem decomposition of a matrix of the same size as the correlation matrix in each step. Since the number of underlying names for CDO structures, and even more so for so-called CDO^2 derivatives, can be rather large indeed, the iterative procedure can be comparatively time consuming, even when we restrict ourselves to the use of a single common factor. This is primarily due to the fact that the computational expense of the eigensystem calculation of a matrix of size $n \times n$ grows like $\mathcal{O}\left(n^3\right)$.

The iterative procedure shown in [ASB03] can be summarised as follows. We wish to find a matrix $A \in \mathbb{R}^{n \times m}$ with $m < n$ such that the objective function given by the Frobenius norm of the difference between the target correlation matrix C and $AA^\top$, i.e.

$$
\begin{aligned}
\chi_{\mathrm{Frobenius}} &= \operatorname{Tr}\left[\left(C - AA^\top\right)\left(C - AA^\top\right)^\top\right] \\
&= \textstyle\sum_{i,j,i\neq j}\left(c_{ij} - \sum_{k=1}^m a_{ik}a_{jk}\right)^2 , \quad (2)
\end{aligned}
$$

is minimal. Starting with an arbitrary diagonal matrix $F^{(0)} \in \mathbb{R}^n$, for instance by setting all entries of $F^{(0)}$ to zero, we iterate

- Find the eigensystem $S^{(i)}$ and diagonal eigenvalue matrix $\Lambda^{(i)}$ of $\left(C - F^{(i)}\right)$ such that

$$C - F^{(i)} = S^{(i)} \cdot \Lambda^{(i)} \cdot S^{(i)}{}^\top \; . \tag{3}$$

- Set

$$A^{(i)} := S^{(i)} \cdot \sqrt{\Lambda_m^{(i)}} \tag{4}$$

where $\Lambda_m^{(i)}$ is constructed from $\Lambda^{(i)}$ by overriding all but the m largest eigenvalue entries with 0.

- Set

$$F^{(i)} := \mathbf{1} - \operatorname{diag}\left[A^{(i)} A^{(i)}{}^\top\right] \; . \tag{5}$$

until $||F^{(i+1)} - F^{(i)}||$ is sufficiently small.

It is noteworthy to observe that the iterative projection method above does not guarantee that the convergence values of the matrix A has row vectors that are all less than or equal to one in their Euclidean norm. In fact, a simple example such as the matrix

$$
\begin{pmatrix}
1 & 1/5 & 2/5 \\
1/5 & 1 & 3/5 \\
2/5 & 3/5 & 1
\end{pmatrix}
\tag{6}
$$

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that is to be projected onto a single column weighting matrix ($m = 1$) will already lead to one of the three loading factors to be greater than one. Whenever this happens, typically, the globally optimal fit satisfying the constraint that the Euclidean norms of the row vectors of A are less than or equal to 1, has at least one of the row vectors of norm exactly 1. In that case, the iterative procedure must be stopped in the $(i + 1)$-th step if the largest row vector norm of $A^{(i+1)}$ exceeds 1, i.e. if $\max_{j=1...m} \sum_{k=1}^{m} (a_{jk}^{(i+1)})^2 > 1$. In practice, it then usually suffices to use a linear interpolation between $A^{(i)}$ and $A^{(i+1)}$ such that the longest row vector(s) of the interpolated matrix is(are) exactly of unit length. This can be done by setting

$$\alpha_j := \sum_{k=1}^{m} (a_{jk}^{(i+1)})^2 \qquad \beta_j := \sum_{k=1}^{m} a_{jk}^{(i+1)} a_{jk}^{(i)} \quad (7)$$

$$\gamma_j := \sum_{k=1}^{m} (a_{jk}^{(i)})^2 \qquad r_j := \alpha_j - 2\beta_j + \gamma_j \quad (8)$$

$$p_j := 2\frac{\beta_j - \gamma_j}{r_j} \qquad q_j := \frac{\gamma_j - 1}{r_j} \qquad (9)$$

$$s_j := \sqrt{\frac{1}{4}p_j^2 - q_j} - \frac{1}{2}p_j \qquad s := \min_{j=1...m} s_j \qquad (10)$$

and using the interpolation

$$A = s \cdot A^{(i+1)} + (1 - s) \cdot A^{(i)} \qquad (11)$$

as the final result of the iteration.

In the very common special case that we seek a single common factor weighting vector, i.e. $m = 1$, there is a remarkably simple alternative to construct an approximate common factor loading vector. In fact, in this situation, we can find a one-factor decomposition vector $\boldsymbol{a}$ *analytically* if we use a different penalty function. Instead of the Frobenius norm of the difference between the target correlation matrix C and $\boldsymbol{a}\boldsymbol{a}^\top$, we use the logarithmic objective function

$$\chi_{\text{Logarithmic Frobenius}} = \sum_{i,j,i\neq j} (\ln c_{ij} - \ln a_i a_j)^2 . \quad (12)$$

The minimisation condition

$$\partial_{a_k} \chi_{\text{Logarithmic Frobenius}} = 0 \qquad (13)$$

for all $k = 1..n$ then leads to the linear system

$$(n - 1)\ln a_k + \sum_{i\neq k} \ln a_i = \kappa_k \qquad (14)$$

with

$$\kappa_k := \sum_{i\neq k} \ln c_{ik} . \qquad (15)$$

The solution of (14) is

$$\ln a_k = \frac{1}{n-2}\left(\kappa_k - \frac{\sum_{i=1}^{n}\kappa_i}{2(n-1)}\right) \qquad (16)$$

which presents our main result. In numerical tests with realistic correlation matrices, we found that the constructive formula (16) produces correlation matrix approximations that are remarkably close to those one can obtain from the iterative approach given in [ASB03] in the one-factor case. Unlike the iterative method necessary when we use the Frobenius norm, though, the computational effort does not grow like $\mathcal{O}(n^3)$, but only like $\mathcal{O}(n^2)$ which makes a major difference for baskets with $n \approx 300$–500 or even higher.

The reader may have noticed that there is no general guarantee that the solution given by (16) produces a_k that are all less than or equal to one. A violation of the requirement $a_k \leq 1$ is, though, for large[1] n, only possible when the square[2] of the geometric average of the entries of one row of the correlation matrix exceeds the geometric average of all the correlation coefficients. Since this condition is likely to clash with the fact that the correlation matrix has to be positive semi-definite, it is rarely incurred in practice. The solution (16) does, however, not allow for negative a_k which is usually desirable in the context of basket credit derivatives pricing.

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[1] For small n, it suffers the same problems as the iterative projection method. In fact, for the example (6), it gives a solution with $a_3 > 1$ that is almost identical to the convergence solution of the iterative projection method without the early termination (11).

[2] actually, the $(2 \cdot \frac{n-1}{n})$-th power which is almost the square for large n