The Link between Caplet and Swaption Volatilities in a BGM/J Framework: Approximate Solutions and Empirical Evidence

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Abstract

We present an approximation for the volatility of European swaptions in a forward rate based Brace-Gatarek-Musiela/Jamshidian framework [BGM97, Jam97] which enables us to calculate prices for swaptions without the need for Monte Carlo simulations. Also, we explain the mechanism behind the remarkable accuracy of these approximate prices. For cases where the yield curve varies noticeably as a function of maturity, a second, and even more accurate formula is derived.

1 Introduction and motivation

In a forward-rate based BGM/J [BGM97, Jam97] approach, once the time-dependent instantaneous volatilities and correlations of the forward rates have been specified, their stochastic evolution is completely determined. Since swap rates are linear combinations (with stochastic weights) of forward rates, it follows that their dynamics are also fully determined once the volatilities and correlations of the forward rates have been specified. Some (very rare) complex derivatives depend exclusively on the volatility of either set of state variables (forward or swap rates). In general, one set of rates dominate the value of a given product, but the other set still contribute to a significant extent. Trigger swaps are a classic example of a product where the relative location of the strike and the barrier level can radically shift the relative importance of forward and swap rates. In practical applications it is therefore extremely important to ascertain the implications for the dynamics of the swap rates, given a particular choice of dynamics for the forward rates and *vice versa*. Unfortunately, as shown later on, the *correct* evaluation of the swaption prices implied by a choice of forward rate volatilities and correlations is a

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conceptually straightforward but computationally intensive task. This note therefore presents two simple but very effective approximations which allow the estimation of a full swaption volatility matrix in a fraction of a second.

The formula that contains one of the results of the present paper, expression (17), has recently appeared, albeit in a different form, in the literature (Jamshidian [Jam97], Andersen and Andreasen [AA00], and Hull and White [HW00]). Our work, however, also provides a financial interpretation of the reason why the formula works so well, by decomposing the contributions arising from parallel curve shifts and remainder terms, which can be readily intuitively understood. Furthermore, our approach splits the approximation into a term which is always present and a second (shape correction) term which will be non-zero only in the presence of non-flat yield curves, further enhancing the intuitive grasp of the approximation. Finally, we present the results of the numerical experiments performed not only for stylized yield/volatility structures, but also in the market-realistic case of non-flat volatility structures, very long maturities (20 years), and using real market data.

2 Statement of the problem

The framework we are interested in is that of a standard LIBOR market model with finite-tenor forward rates as described, for instance, in Jamshidian [Jam97], Musiela and Rutkowski [MR97], or any of [Reb98, Reb99, Jäc02]. In the pricing measure, Q, associated with one of the available numéraires (discount bonds) we can write for the dynamics of each forward rate:

$$\frac{df_i}{f_i} = \mu_i(f_i, t)dt + \sigma_i(t)d\widetilde{W}_i \tag{1}$$

where $d\widetilde{W}_i$ are the increments of standard Q-Wiener processes. The no-arbitrage evolution of the forward rates is specified by the choice of a particular functional form for the forward-rate instantaneous volatilities and for the forward-rate/forward-rate correlation function. The drift coefficients in equation (1) result from the usual martingale requirement

$$E^{(\mathcal{N})}\left[\frac{f_i P_{i+1}}{\mathcal{N}}\right] = \frac{f_i(0)P_{i+1}(0)}{\mathcal{N}(0)} \ , \tag{2}$$

i.e. from the condition that the expectation of a cashflow in units of a numéraire asset $\mathcal{N}$ (in the associated measure) must be equal to the value of that ratio at inception (P_{i+1} denotes the zero coupon bond maturing at the payment time of the i-th forward rate f_i).

Let $\sigma^{N\times M}(t)$ denote the relative instantaneous volatility at time t of a swap rate $SR^{N\times M}$ expiring N years from today and maturing M years thereafter. This swap rate can be viewed as depending on the forward rates of that part of the yield curve in an approximately linear way, namely

$$SR^{N\times M}(t) = \sum_{i=1}^{n} w_i f_i(t) \ , \tag{3}$$

with the weights w_i given by

$$w_i = \frac{P_{i+1}\tau_i}{\sum_{k=1}^{n} P_{k+1}\tau_k} \ . \tag{4}$$

In equation (4), τ_i is the associated accrual factor such that

$$P_{i+1} = \left[\prod_{k=1}^{i}(1 + f_k\tau_k)\right]^{-1} \cdot P_1 \ , \tag{5}$$

n is the number of forward rates in the swap as illustrated schematically in figure 1, and we have

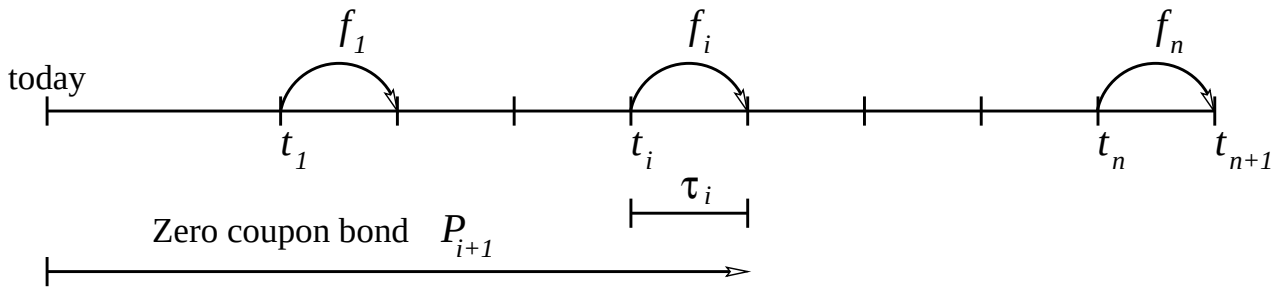

Figure 1: The forward rates determining the individual payments of the swap.

identified $t_1 := N$ and $t_{n+1} := N + M$. If we employ the approximation that the weights $\{\boldsymbol{w}\}$ are effectively constant and thus independent of the forward rates[1], we arrive, with the aid ofItô's lemma, at the equation

$$\left[\sigma^{N\times M}\right]^2 \quad = \quad \frac{\sum_{j,k=1}^{n} w_j w_k f_j f_k \rho_{jk}\sigma_j\sigma_k}{\left[\sum_{i=1}^{n} w_i f_i\right]^2} \tag{6}$$

where dependence on time has been omitted for clarity and, as usual, $\sigma_j(t)$ is the time-t instantaneous volatility of forward rate f_j, and $\rho_{jk}(t)$ is the instantaneous correlation between forward rate f_j and f_k.

Expression (6) shows that the instantaneous volatility at time $t > 0$ of a swap rate is a stochastic quantity, depending on the coefficients $\{\boldsymbol{w}\}$, and on the future realization of the forward rates underlying the swap rate $\{\boldsymbol{f}\}$. One therefore reaches the conclusion that, starting from a purely deterministic function of time for the instantaneous volatilities of the forward rates, one arrives at a rather complex, and *stochastic*, expression for the instantaneous volatility of the corresponding swap rate. Therefore, in order to obtain the price of a European swaption corresponding to a given choice of forward-rate instantaneous volatilities, one is faced with a computationally rather cumbersome task: to begin with, in order to obtain the total Black volatility of a given European swaption to expiry, in fact, one first has to integrate its swap-rate instantaneous volatility

$$\left[\sigma^{N\times M}_{\text{Black}}\right]^2 \cdot t_1 \quad = \quad \int_{u=0}^{t_1} \left[\sigma^{N\times M}(u)\right]^2 du \tag{7}$$

[1] An accurate discussion of the dependence of the swap rates as given in equation (3) on the forward rates will be given in section 3.

with t_1 being the time horizon of expiry of the option in N years from today as defined before.

As equation (6) shows, however, at any time u there is one (different) swap rate instantaneous volatility for *any* future realization of the forward rates from today to time u. But since every path gives rise to a particular swap-rate instantaneous volatility via the dependence on the path of the quantities $\{\boldsymbol{w}\}$ and $\{\boldsymbol{f}\}$, there seems to be no such thing as a *single* unique total Black volatility for the swap rate. Rather, if one starts from a description of the dynamics of forward rates in terms of a deterministic volatility, there appears to be one Black volatility for a given swap rate associated with each and every realization of the forward rates along the path of the integral. Notice that the implications of equation (6) are farther-reaching than the usual (and correct) claim that log-normal forward rates are incompatible with log-normal swap rates[2]. By equation (6) one can conclude that, starting from a purely *deterministic* volatility for the (logarithm of) the forward rates, the instantaneous volatility of the corresponding swap rate is a stochastic quantity, and that the quantity $\int_{u=0}^{t_1} \left[\sigma^{N\times M}(u)\right]^2 du$ is a path-dependent integral that cannot be equated to the (path-independent) real number $\left[\sigma_{\text{Black}}^{N\times M}\right]^2 \cdot t_1$. Calculating the value of several European swaptions, or, perhaps, of the whole swaption matrix, therefore becomes a very burdensome task, the more so if the coefficients of the forward-rate instantaneous volatilities are not given *a priori* but are to be optimised via a numerical search procedure so as to produce, say, the best possible fit to the swaption market.

Some very simple but useful approximations are however possible. In order to gain some insight into the structure of equation (6), one can begin by regarding it as a weighted average of the products $\rho_{jk}(t)\sigma_j(t)\sigma_k(t)$ with doubly-indexed coefficients $\zeta_{jk}(t)$ given by

$$\zeta_{jk}(t) \quad = \quad \frac{w_j(t)f_j(t)\, w_k(t)f_k(t)}{\left[\sum_{i=1}^{n} w_i(t)f_i(t)\right]^2} \; . \tag{8}$$

This turns equation (6) into

$$\left[\sigma^{N\times M}(t)\right]^2 \quad = \quad \sum_{j,k=1}^{n} \zeta_{jk}(t)\rho_{jk}(t)\sigma_j(t)\sigma_k(t) \quad . \tag{9}$$

For a given point in time, and for a given realization of the forward rates, these coefficients are, in general, far from constant or deterministic. Their stochastic evolution is fully determined by the evolution of the forward rates. One can, however, distinguish two important cases: the first (case 1) refers to (proportionally) parallel moves in the yield curve; the second (case 2) occurs when the yield curve experiences more complex changes. Intuitively, one can therefore regard the results presented in the following as pertaining to movements of the yield curve under shocks of the first principal component[3] for case 1, or to higher principal components for case 2. For the purpose of the discussion to follow, it is also important to keep in mind the typical relative magnitude of the first principal component shocks relative to the higher modes of deformation.

[2] See [Reb99] for a discussion of the price implications of the joint log-normal assumptions.

[3] Since, as shown below, the coefficients are approximately constant for identical proportional changes in the forward rates, the principal components should be thought of as referring to log changes.

With this distinction in mind, one can notice that in the first (parallel) case each individual weight is only mildly dependent on the stochastic realization of the forward rate at time t. Intuitively this can be understood by observing that a given forward rate occurs both in the numerator and in the denominator of equation (8). So the effects on the coefficients of a (reasonably small) identical proportional change in the forward rates to a large extent cancel out. This is shown in figures 2, 3, and 4 for the particular case study illustrated in table 1. The first of the three figures shows the changes to which the yield curve

i	t_i	P_i	f_i	w_i
	0.00	1.000000	5.278%	
	0.25	0.986976	5.747%	
	0.50	0.972996	6.178%	
	0.75	0.958196	6.572%	
	1.00	0.942707	6.928%	
	1.25	0.926657	7.246%	
	1.50	0.910169	7.526%	
	1.75	0.893360	7.768%	
1	2.00	0.876340	7.972%	0.072409
2	2.25	0.859220	8.138%	0.070965
3	2.50	0.842080	8.265%	0.069529
4	2.75	0.825040	8.355%	0.068106
5	3.00	0.808160	8.406%	0.066704
6	3.25	0.791520	8.419%	0.065329
7	3.50	0.775210	8.393%	0.063987
8	3.75	0.759280	8.329%	0.062681
9	4.00	0.743790	8.227%	0.061418
10	4.25	0.728800	8.087%	0.060201
11	4.50	0.714360	7.908%	0.059034
12	4.75	0.700510	7.692%	0.057920
13	5.00	0.687290	7.437%	0.056863
14	5.25	0.674740	7.144%	0.055865
15	5.50	0.662900	6.814%	0.054929
16	5.75	0.651800	6.445%	0.054058
17	6.00	0.641470		

Table 1: The initial yield curve and the column of weights $\{\boldsymbol{w}\}$ for the 2×4-quarterly swaption.

was subjected (rigid up and down shifts by 25 basis points); figure 3 displays the percentage changes in the coefficients $\{\boldsymbol{\zeta}\}$ for the longest co-terminal swap in moving from the initial yield curve to the yield curve shocked upwards by 25 basis points; figure 4 then shows the average of the percentage changes in the coefficients $\{\boldsymbol{\zeta}\}$ corresponding to the equi-probable up and down 25 basis point shifts.

For more complex changes in the shape of the yield curve, the individual coefficients remain less and less constant with increasing order of the principal component. In the less benign case of tilts and bends in the forward curve, the difference between the coefficients calculated with the initial values of the forward rates and after the yield curve move will in general be significant. However, in these cases one observes that the *average* of each individual weight corresponding to a positive and negative move

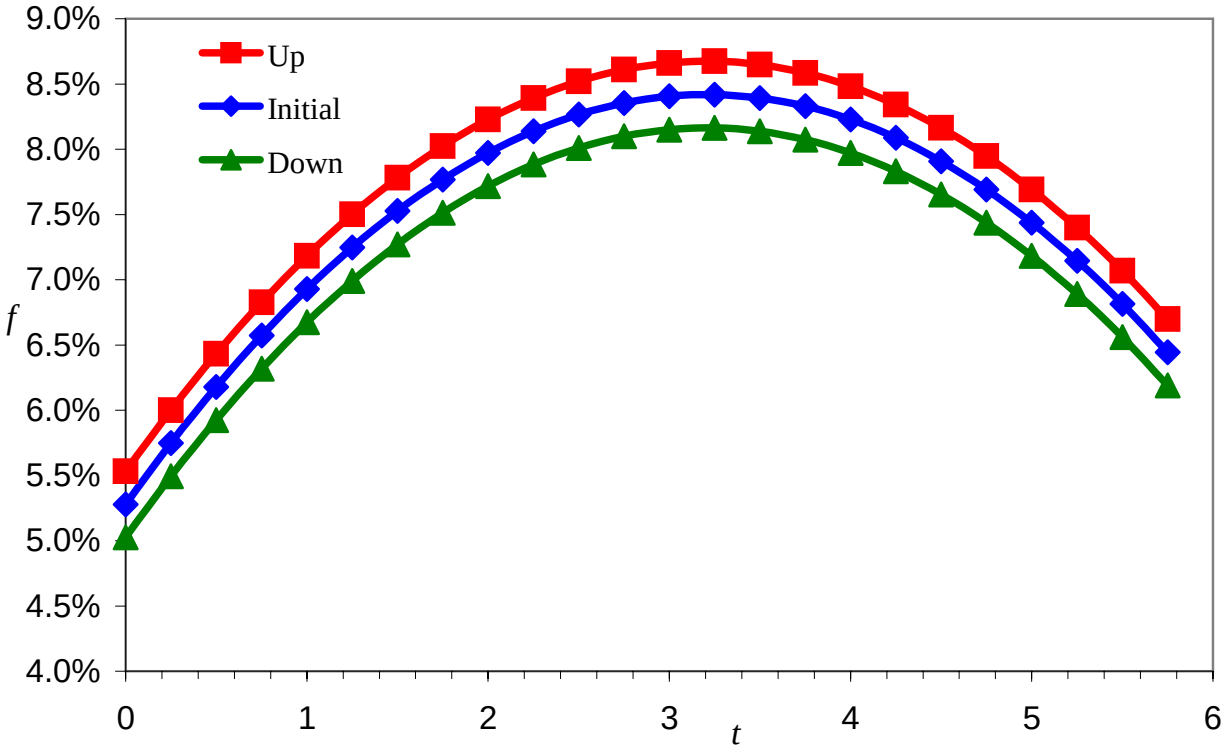

Figure 2: Yield curves (rigid shift)

of the same magnitude (clockwise and counter-clockwise tilt, increased and decreased curvature) is still remarkably constant. This feature, needless to say, is even more marked for the parallel movement, as shown in figure 4. On the basis of these observations we are therefore in a position to reach two simple but useful conclusions:

1. To the extent that the movements in the forward curve are dominated by a first (parallel) principal component, the coefficients $\{\zeta\}$ are only very mildly dependent on the path realizations.

2. Even if higher principal components are allowed to shock the forward curve, the *expectation* of the future swap rate instantaneous volatility is very close to the value obtainable by using today's values for the coefficients $\{\zeta\}$ and the forward rates $\{\boldsymbol{f}\}$.

Note that the second statement has wider applicability (it does not require that the forward curve should only move in parallel), but yields weaker results, only referring as it does to the *average* of the instantaneous volatility. Note also that the average of the weights over symmetric shocks becomes less and less equal to the original weights as the complexity of the deformation increases; on the other hand we know that relatively few principal components can describe the yield curve dynamics to a high degree of accuracy. Therefore, the negative impact of a progressively poorer approximation becomes correspondingly smaller and smaller.

Given the linearity of the integration operator which allows the user to move from instantaneous volatilities to Black volatilities via equation (7), the second conclusion can be transferred to these latter quantities, and it can therefore be re-stated as:

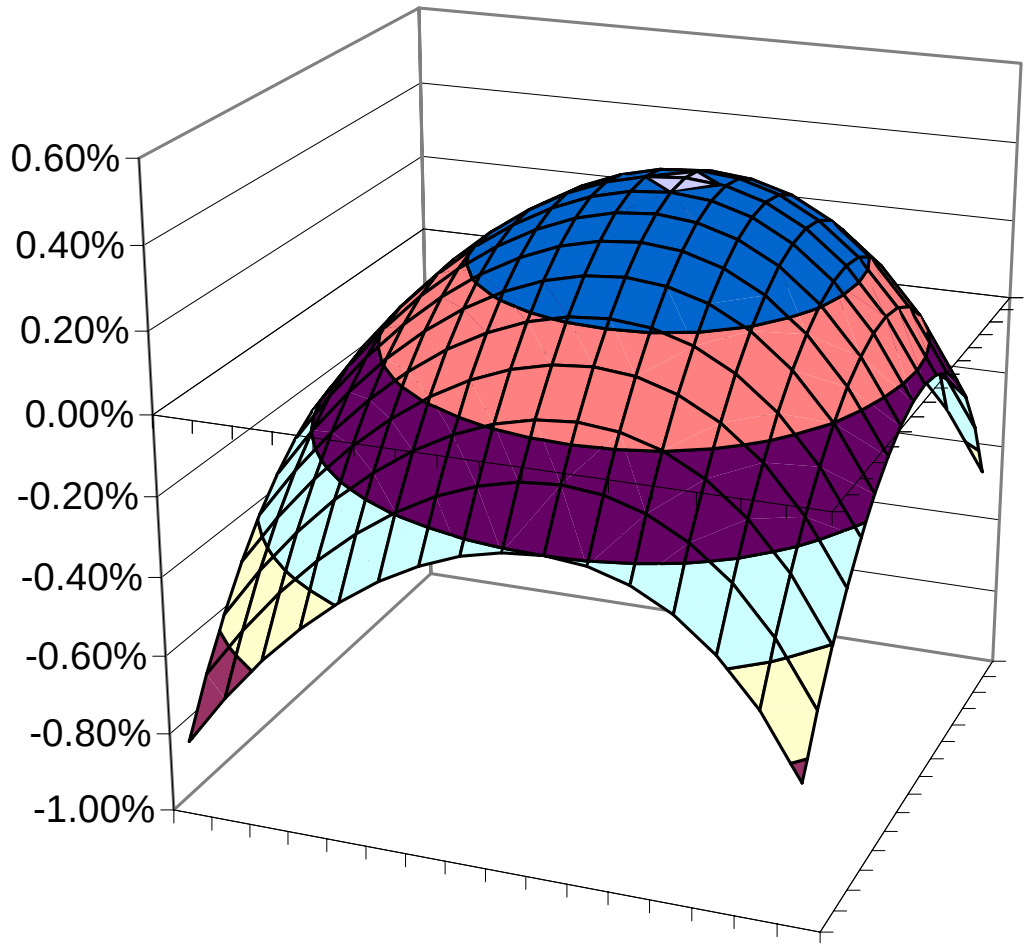

Figure 3: Percentage differences in the elements of the coefficients $\boldsymbol{\zeta}$ caused by a 25 basis points upward shift of the yield curve

> Even if higher principal components are allowed to shock the forward curve, the expectation of *the average Black volatility* is very close to the value obtainable by integrating the swap rate instantaneous volatilities calculated using today's values for the coefficients $\{\boldsymbol{\zeta}\}$ and the forward rates $\{\boldsymbol{f}\}$.

It is well known, on the other hand, that the price of an at-the-money plain-vanilla option, such as a European swaption, is to a very good approximation a linear function of its implied Black volatility[4]. Therefore it follows that, as long as one includes in the average all possible changes in shape of the forward curve in a symmetric fashion, the resulting average of the prices for the European swaption under study obtained using the different weights, $\zeta_{jk}(t)$, will be very close to the single price obtained using the current values for $\{\boldsymbol{\zeta}\}$ and $\{\boldsymbol{f}\}$.

This conclusion, by itself, would not be sufficient to authorize the trader to quote as *the* price for the European swaption the (approximate) average over the price distribution. More precisely, the situation faced by a trader who uses a forward-rate-based implementation for pricing and hedging is as follows: starting from a dynamics for forward rates described by a deterministic volatility, he arrives at

[4] See e.g. [LS95].

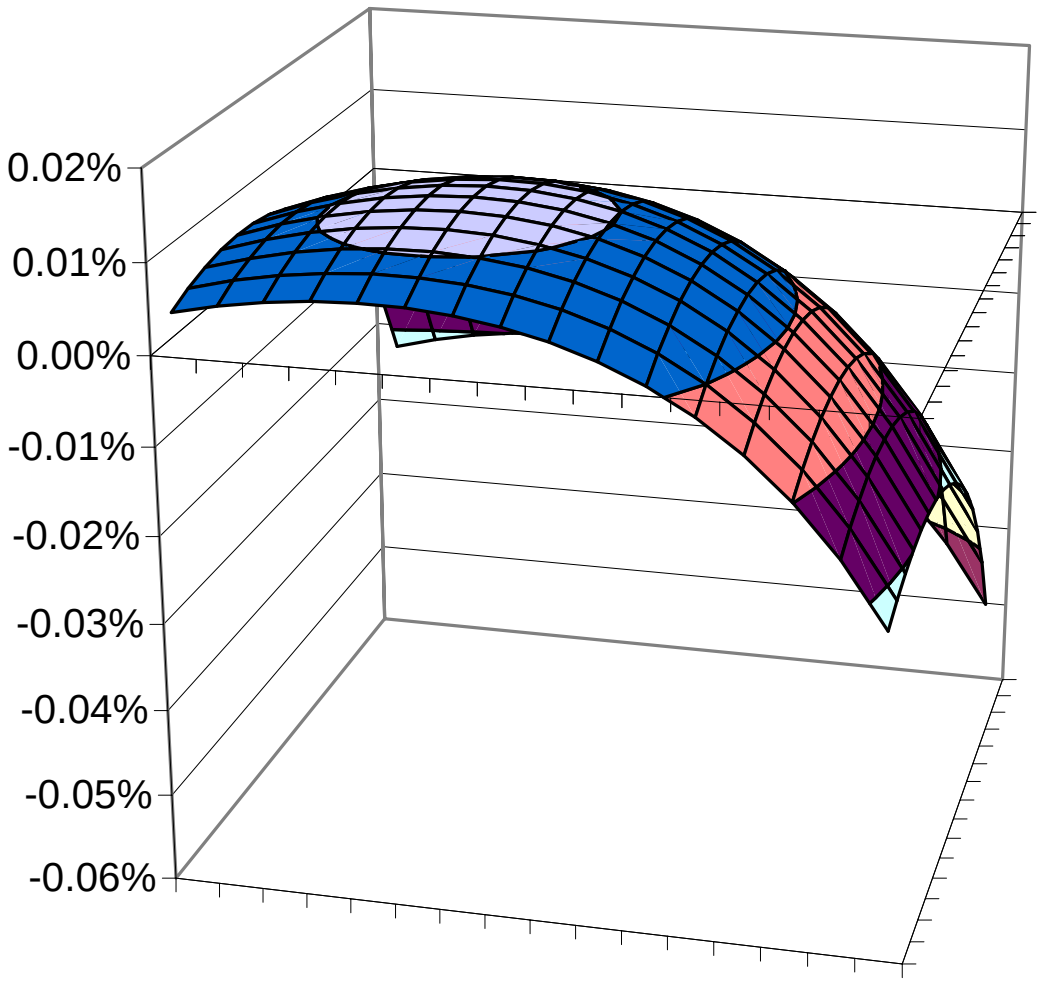

Figure 4: Average percentage differences in the elements of the coefficients ζ caused by equi-probable 25 basis points upward and downward shifts of the yield curve

a distribution of swaption prices. Therefore, by engaging in a self-financing trading strategy in forward rates to hedge a swaption assunimg i) that both sets of quantities are log-normally distributed and ii) that. their volatilities are simultaneously deterministic, he will not, in general, manage to produce an exact replication of the swaption payoff by its expiry; therefore the combined portfolio (swaption plus dynamically re-hedged holdings of forward rates) will have a finite variance (and higher moments) at expiry. Strictly speaking, the risk-averse trader will therefore not make a price simply by averaging over the final portfolio outcomes. The dispersion of the swaption prices around their average is however very small. If one therefore assumes that swaptions and forward rates can have simultaneously deterministic volatilities, and makes use of the results in [Reb98] about the likely impact of the joint log-normal assumption, it is possible to engage in a trading strategy that will produce, by expiry, imperfect but very good replication. In other terms, the trader who were to associate to the swaption a price significantly different from the average would have to have a utility function exceedingly sensitive indeed to small variations in his final wealth. Therefore statements 1 and 2 together lead one to surmise that the

expression

$$\left[\sigma^{N\times M}(t)\right]^2 \quad \simeq \quad \sum_{j,k=1}^{n} \zeta_{jk}(0)\rho_{jk}(t)\sigma_j(t)\sigma_k(t) \tag{10}$$

should yield a useful approximation to the instantaneous volatility of the swap rate, and, ultimately, to the European swaption price. It is essential to note that the above equation differs subtly but fundamentally from equation (9) in that the coefficients $\{\boldsymbol{\zeta}\}$ are no longer stochastic quantities, but are evaluated using today's known values for the forward rates and discount factors. What's more, by virtue of the previous results on the *average* of the $\boldsymbol{\zeta}$ coefficients, a robust approximation for the equivalent implied Black volatility of a European swaption can be derived since the risk-neutral price of an option is given by the *expectation*, i.e. the *average* over the risk-neutral measure. The expression for the average Black volatility then becomes

$$\sigma_{\text{Black}}^{N\times M} \quad = \quad \sqrt{\sum_{j,k=1}^{n} \zeta_{jk}(0) \int_{u=0}^{1} \sigma_j(ut_1)\sigma_k(ut_1)\rho_{jk}(ut_1)\, du} \quad . \tag{11}$$

Equation (11) should be very useful in the context of calibration of FRA-based BGM/J models to market given European swaption volatilities. It enables us to calculate prices for the whole swaption matrix *without having to carry out a single Monte Carlo simulation* and thus to solve the highly cumbersome problem of calibration with great ease.

As shown in the result section, the quality of this approximation is very good. In those situations (noticeably non-flat yield curves) where it begins to prove unsatisfactory, it can be easily improved upon by a natural extension, which is presented in the next section.

3 Refining the approximation

The application of Itô's lemma to equation (3) gives equation (6) only if one assumes that the weights $\{\boldsymbol{w}\}$ are independent of the forward rates $\{\boldsymbol{f}\}$. More correctly, and neglecting the deterministic part irrelevant for this discussion[5], Itô's lemma gives

$$\begin{aligned} \frac{dSR}{SR} &= \sum_{i=1}^{n} \frac{\partial SR}{\partial f_i} \cdot \frac{df_i}{SR} \\ &= \sum_{i=1}^{n} \frac{\partial SR}{\partial f_i} \cdot \frac{f_i}{SR} \cdot \sigma_i d\widetilde{W}_i \end{aligned} \tag{12}$$

wherein the Wiener processes $\widetilde{W}_i$ are correlated, i.e.

$$\left\langle d\widetilde{W}_i \cdot d\widetilde{W}_j \right\rangle = \rho_{ij} dt \ . \tag{13}$$

[5]The neglected terms are truly irrelevant in the following since they drop out as soon as we calculate instantaneous swap-rate/swap-rate covariances.

Given the definition

$$A_k = \sum_{j=k}^{n} P_{j+1} f_j \tau_j \tag{14}$$

of co-terminal floating-leg values and

$$B_k = \sum_{j=k}^{n} P_{j+1} \tau_j \tag{15}$$

for the co-terminal fixed-leg annuities, we obtain after some algebraic manipulations (see appendix A for a derivation)

$$\frac{\partial SR}{\partial f_i} = \left\{ \frac{P_{i+1}\tau_i}{B_1} - \frac{\tau_i}{1+f_i\tau_i} \cdot \frac{A_i}{B_1} + \frac{\tau_i}{1+f_i\tau_i} \cdot \frac{A_1 B_i}{B_1^2} \right\} . \tag{16}$$

This enables us to calculate the following improved formula for the coefficients $\{\boldsymbol{\zeta}\}$:

$$\zeta_{ij} = \left[\underbrace{\frac{P_{i+1} f_i \tau_i}{A_1}}_{\text{as in equation (8)}} + \underbrace{\frac{(A_1 B_i - A_i B_1) f_i \tau_i}{A_1 B_1 (1+f_i\tau_i)}}_{\text{shape correction}} \right] \cdot \left[\underbrace{\frac{P_{j+1} f_j \tau_j}{A_1}}_{\text{as in equation (8)}} + \underbrace{\frac{(A_1 B_j - A_j B_1) f_j \tau_j}{A_1 B_1 (1+f_j\tau_j)}}_{\text{shape correction}} \right] \tag{17}$$

We call the second term inside the square brackets of equation (17) the *shape correction*. Rewriting this corrective term as

$$\frac{(A_1 B_i - A_i B_1) f_i \tau_i}{A_1 B_1 (1+f_i\tau_i)} = \frac{f_i \tau_i}{A_1 B_1 (1+f_i\tau_i)} \cdot \sum_{l=1}^{i-1} \sum_{m=i}^{n} P_{l+1} P_{m+1} \tau_l \tau_m (f_l - f_m) \tag{18}$$

highlights that it is a weighted average over inhomogeneities of the yield curve. In fact, for a flat yield curve, all of the terms $(f_l - f_m)$ are identically zero and the righ-hand-side of equation (17) is identical to that of equation (8). It should be noted, however, that an adaption of both the constant-weights approximation given by equation (8) and the refined equation (17) to the situation when payments on the floating and the fixed side of the swap have different frequencies results in formulæ that evaluate to different figures even when the initial yield curve is flat [Sch00].

4 Specific functional forms

We are finally in a position to conduct some empirical tests. In order to do so one needs to specify a correlation function ρ_{jk}. In general this function will depend both on calendar time, and on the expiry time of the two forward rates. If one makes the assumptions i) that the correlation function is time homogeneous, and ii) that it only depends on the relative distance in years between the two forward rates in question (i.e. on $|t_j - t_k|$), further simplifications are possible. The expression for the average Black volatility now becomes:

$$\left[\sigma_{\text{Black}}^{N\times M}\right]^2 \cdot t_1 = \sum_{j,k=1}^{n} \zeta_{jk}(0) \cdot \rho_{|t_j - t_k|} \int_{u=0}^{t_1} \sigma_j(u) \sigma_k(u)\, du \tag{19}$$

Focussing then on the instantaneous volatilities, if the simple yet flexible functional form discussed in [Reb99] is adopted, i.e. if $\sigma_j(t)$ is taken to be equal to

$$\sigma_j(t) = k_j \left[(a + b(t_j - t))\, \mathrm{e}^{-c(t_j - t)} + d \right] \tag{20}$$

then the integrals in (19) can be easily carried out analytically (see appendix B), and a whole swaption matrix can be calculated in fractions of a second.

Given the difficulties in estimating reliably and robustly correlation functions (let alone in trying to estimate their possible time dependence), the assumption of time homogeneity for the correlation function is rather appealing. The further assumption that $\rho_{jk} = \rho_{|t_j - t_k|}$, or as in our particular choice

$$\rho_{jk} = \mathrm{e}^{-\beta |t_j - t_k|} \quad \text{with} \quad \beta = 0.1\,, \tag{21}$$

is more difficult to defend on purely econometric grounds: it implies, for instance, that the de-correlation between, say, the front and the second forward rate should be the same as the de-correlation between the ninth and the tenth. Luckily, European swaption prices turn out to be relatively insensitive to the details of the correlation function, and this assumption can be shown to produce in most cases prices very little different from those obtained using more complex and realistic correlation functions.

5 Empirical Results on European Swaptions

The results and the arguments presented so far indicate that it is indeed plausible that the instantaneous volatility of a swap rate might be evaluated with sufficient precision by calculating the stochastic coefficients $\{\boldsymbol{\zeta}\}$ using the initial yield curve. The ultimate proof of the validity of the procedure, however, is obtained by checking actual European swaption prices. The following test was therefore carried out.

- first of all, the instantaneous volatility function described above in equation (20) was used, with parameters chosen as to ensure a realistic and approximately time homogeneous behaviour for the evolution of the term structure of volatilities. This feature was not strictly necessary for the test, but the attempt was made to create as realistic a case study as possible. In particular, the values of the vector $\boldsymbol{k}$ implicitly defined by equation (20) were set to unity, thereby ensuring a time-homogeneous evolution of the term structure of volatilities (see [Reb99] on this point);

- given this parametrised form for the forward-rate instantaneous volatility, the instantaneous volatility of a given swap was integrated out to the expiry of the chosen Euroepan swaption. The correlation amongst the forward rates was assumed to be given by equation (21). The value of this integral could therefore be evaluated analytically and gave the required approximate implied volatility for the chosen European swaption;

- with this implied volatility the corresponding approximate Black price was obtained;

- given the initial yield curve and the chosen instantaneous volatility function for the forward rates, an exact FRA-based BGM/J Monte Carlo evaluation of the chosen European swaption price was carried out. By 'exact' we mean that, for this evaluation, the same correlation function was used in the estimation of the approximate price, and by retaining as many stochastic driving factors as forward rates in the problem. This meant retaining up to 40 factors for the case studies presented below. In order to ensure that the Monte Carlo results were highly accurate, the following comparative tests were conducted:-

  – Each simulation was repeated with a variety of number generators taken from [PTVF92], with an increasing number of simulations up to 131072 paths per evaluation (the residual Monte Carlo error is below the visual resolution of the diagrams in figures 5 and 6);

  – The stochastic differential equation governing the dynamics of the underlying forward rates was integrated using a predictor-corrector drift method [HJJ01] and a time-discretisation of 0.25 years per step out until expiry of the swaption (we chose the zero coupon bond maturing on the last reset time of the underlying swap as numéraire);

  – For comparison, the standard log-Euler drift method was also employed which showed that for the time discretisation of 0.25 years per step both the conventional log-Euler Monte Carlo method and the predictor-corrector approach resulted in at-the-money swaption prices that are within less than 0.2 basis points of each other;

  – the values of the swaps and FRAs that can be obtained as a by-product of the procedure were calculated separately to check against the possible presence of drift biases. The results, (not shown in the tables below) indicated discrepancies from the swap and forward rates always less than 0.1 basis points with respect to the corresponding reference rate;

  Due to all of these safeguards, we were certain that the Monte Carlo results are within 0.2 basis points of the continuous limit;

- the price for the European swaption obtained from the simulation, and the corresponding price obtained using the approximate equation (19) was then compared.

The results are shown in table 2, where we give the discount factors and the resulting prices for at-the-money European swaptions resulting from the different formulas (8) and (17) for a flat yield curve at 7% (annually compounded) and a GBP yield curve for August 10$^{\text{th}}$, 2000. The volatilities of the forward rates were modelled according to equation (20) with $a = -5\%$, $b = 0.5$, $c = 1.5$, and $d = 15\%$. Correlation was assumed to be as in equation (21) with $\beta = 0.1$. For each yield curve, all of 40 co-terminal semi-annual at-the-money swaptions with maturity of the final payment after $20\,{}^1\!/_2$ years were calculated. The pricing errors for the different approximations are also shown in figures 5 and 6 for the flat and the GBP yield curve, respectively. As expected, the two approximations are identical for a flat yield curve. For the GBP yield curve, the approximation (8) diverges from the exact price up to 10 basis points for swaptions of expiry around 3 years. The shape corrected formula

	Flat at 7% annually					GBP for August 10th, 2000				
Time [years]	discount factor	Price from eq. (8)	Price from eq. (17)	Monte Carlo price	Vega	discount factor	Price from eq. (8)	Price from eq. (17)	Monte Carlo price	Vega
0.5	0.966736	2.572%	2.572%	2.569%	0.202%	0.969514	2.418%	2.474%	2.472%	0.189%
1.0	0.934579	3.475%	3.475%	3.469%	0.273%	0.939441	3.268%	3.344%	3.340%	0.254%
1.5	0.903492	4.042%	4.042%	4.033%	0.318%	0.909913	3.798%	3.887%	3.880%	0.297%
2.0	0.873439	4.427%	4.427%	4.416%	0.350%	0.881024	4.153%	4.250%	4.241%	0.326%
2.5	0.844385	4.699%	4.699%	4.685%	0.373%	0.852807	4.396%	4.497%	4.487%	0.347%
3.0	0.816298	4.890%	4.890%	4.874%	0.389%	0.825482	4.561%	4.664%	4.653%	0.360%
3.5	0.789145	5.022%	5.022%	5.004%	0.400%	0.799100	4.671%	4.772%	4.760%	0.369%
4.0	0.762895	5.107%	5.107%	5.088%	0.406%	0.773438	4.733%	4.832%	4.819%	0.374%
4.5	0.737519	5.154%	5.154%	5.133%	0.409%	0.749042	4.765%	4.861%	4.848%	0.376%
5.0	0.712986	5.169%	5.169%	5.147%	0.409%	0.725408	4.765%	4.858%	4.844%	0.375%
5.5	0.689270	5.157%	5.157%	5.134%	0.406%	0.702527	4.739%	4.827%	4.812%	0.372%
6.0	0.666342	5.122%	5.122%	5.098%	0.402%	0.680361	4.689%	4.772%	4.757%	0.366%
6.5	0.644177	5.065%	5.065%	5.042%	0.395%	0.659402	4.627%	4.705%	4.691%	0.360%
7.0	0.622750	4.991%	4.991%	4.967%	0.388%	0.639171	4.549%	4.622%	4.608%	0.352%
7.5	0.602035	4.901%	4.901%	4.877%	0.378%	0.619580	4.454%	4.523%	4.509%	0.343%
8.0	0.582009	4.796%	4.796%	4.773%	0.368%	0.600668	4.347%	4.410%	4.397%	0.333%
8.5	0.562649	4.679%	4.679%	4.656%	0.357%	0.582455	4.229%	4.287%	4.275%	0.322%
9.0	0.543934	4.550%	4.550%	4.528%	0.345%	0.564873	4.101%	4.154%	4.142%	0.310%
9.5	0.525841	4.411%	4.411%	4.390%	0.332%	0.547888	3.964%	4.011%	4.000%	0.298%
10.0	0.508349	4.263%	4.263%	4.243%	0.319%	0.531492	3.818%	3.860%	3.850%	0.285%
10.5	0.491440	4.107%	4.107%	4.087%	0.305%	0.515651	3.665%	3.702%	3.692%	0.271%
11.0	0.475093	3.943%	3.943%	3.925%	0.290%	0.500360	3.505%	3.537%	3.528%	0.258%
11.5	0.459290	3.773%	3.773%	3.756%	0.276%	0.485543	3.339%	3.365%	3.357%	0.244%
12.0	0.444012	3.596%	3.596%	3.580%	0.261%	0.471240	3.167%	3.188%	3.181%	0.229%
12.5	0.429243	3.414%	3.414%	3.400%	0.245%	0.457861	2.999%	3.017%	3.011%	0.215%
13.0	0.414964	3.227%	3.227%	3.214%	0.230%	0.444977	2.828%	2.842%	2.837%	0.201%
13.5	0.401161	3.035%	3.035%	3.024%	0.214%	0.432554	2.653%	2.665%	2.660%	0.187%
14.0	0.387817	2.840%	2.840%	2.830%	0.199%	0.420575	2.475%	2.484%	2.480%	0.173%
14.5	0.374917	2.640%	2.640%	2.632%	0.183%	0.409019	2.294%	2.301%	2.298%	0.159%
15.0	0.362446	2.438%	2.438%	2.430%	0.167%	0.397888	2.112%	2.116%	2.113%	0.145%
15.5	0.350390	2.232%	2.232%	2.226%	0.151%	0.387341	1.931%	1.934%	1.932%	0.131%
16.0	0.338735	2.023%	2.023%	2.018%	0.136%	0.377196	1.749%	1.751%	1.749%	0.117%
16.5	0.327467	1.811%	1.811%	1.808%	0.120%	0.367435	1.565%	1.567%	1.565%	0.104%
17.0	0.316574	1.597%	1.597%	1.594%	0.104%	0.358056	1.381%	1.382%	1.381%	0.090%
17.5	0.306044	1.381%	1.381%	1.379%	0.089%	0.348978	1.195%	1.195%	1.195%	0.077%
18.0	0.295864	1.161%	1.161%	1.160%	0.074%	0.340292	1.008%	1.008%	1.008%	0.064%
18.5	0.286022	0.938%	0.938%	0.938%	0.059%	0.331614	0.811%	0.811%	0.811%	0.051%
19.0	0.276508	0.712%	0.712%	0.711%	0.044%	0.323265	0.612%	0.611%	0.611%	0.038%
19.5	0.267311	0.480%	0.480%	0.479%	0.029%	0.315460	0.415%	0.415%	0.415%	0.025%
20.0	0.258419	0.242%	0.242%	0.242%	0.014%	0.307945	0.215%	0.215%	0.215%	0.013%
20.5	0.249823					0.300321				

Table 2: The discount factors and results for the two yield curves used in the tests.

(17), however, stays within 1.5 basis points of the correct price which demonstrates how powerful the formula is for the purpose of calibration to European swaptions.

It is also interesting to compare the accuracy obtained by the approximations reported above with the accuracy of the calibration to the swaption matrix. Clearly, there is no single calibration procedure to the swaption matrix, and, if the user were ready to make use of a non-parametric approach whereby the individual covariance elements were used as 'free parameters' a perfect fit could be easily obtained. A more interesting question is how good the fit to the swaption matrix is, if the user employs a reason-

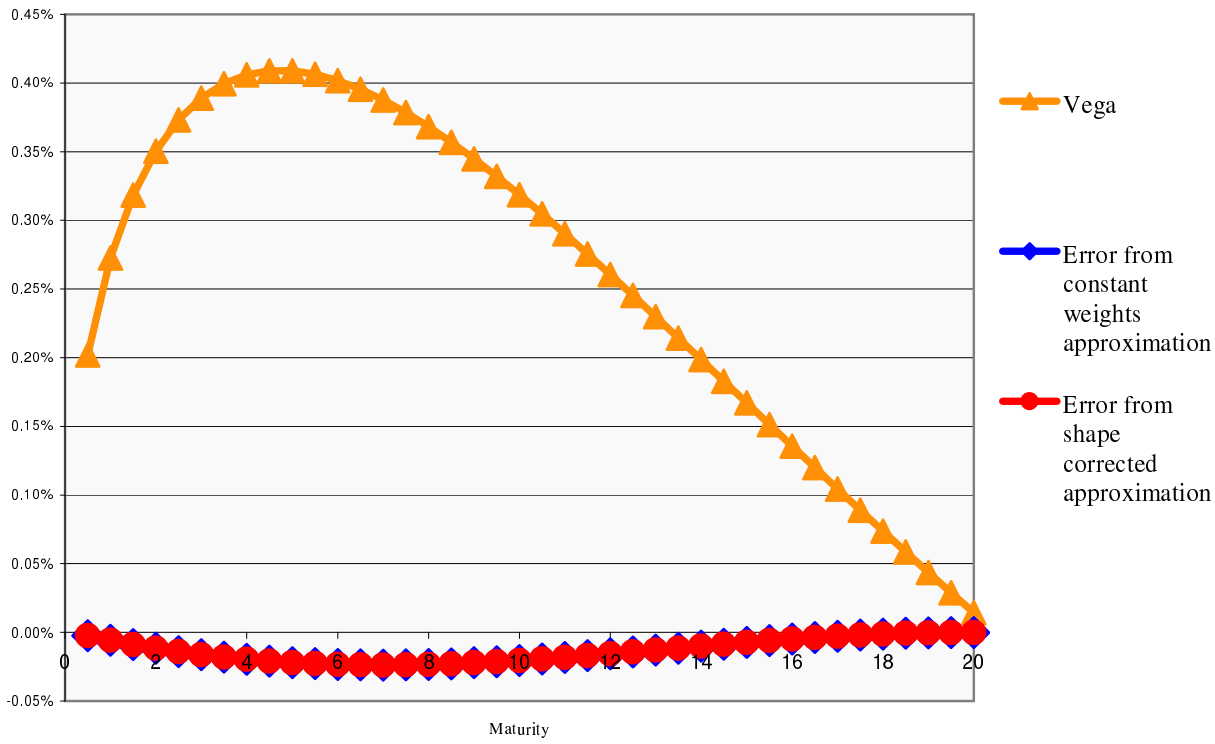

Figure 5: The pricing error from the constant-weights approximation given by equation (8) and including the shape correction as in equation (17) for a flat yield curve at 7% annual yield.

ably parsimonious model, and imposes criteria of financial plausibility. In this context the requirement of approximate time homogeneity for the swaption matrix is particularly appealing, as argued, e.g., by Longstaff, Santa-Clara, and Schwartz [LSCS99, LSCS00]. When this is done they find that the discrepancies between the caplet and the swaption markets are substantial (ranging from approximately 8% to 23% in percentage price error for caps of various maturities), and therefore much larger than the errors entailed by the procedure proposed in this paper. We found very similar results, especially in the post-1998 period.

6 Conclusion

In this article we have derived and analysed a comparatively simple formula for the pricing of European swaptions in the BGM/J framework based on log-normally evolving forward rates. The main result is that a simple summation over weighted FRA/FRA covariances gives a very good approximation for the total equivalent variance incurred by the swap rate. Using the implied volatility equivalent to the total variance in a log-normal option formula, i.e. the Black formula, then suffices to price European swaptions with a remarkable degree of accuracy. We also explained the mechanism responsible for this surprisingly good match between a log-normal pricing formula using an approximate equivalent volatility and a full blown Monte Carlo simulation. The key here was that on average, the weighting coefficients depend only very little on the evolved yield curve for the lower modes of possible

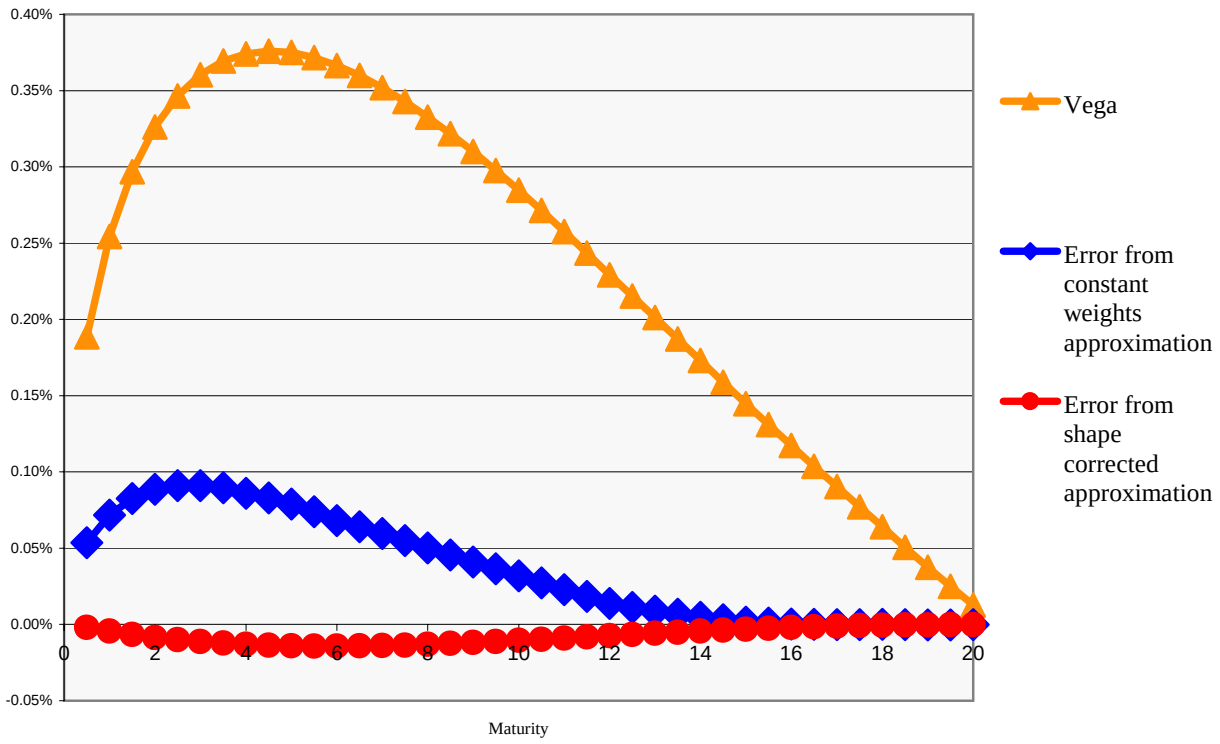

Figure 6: The pricing error from the constant-weights approximation given by equation (8) and including the shape correction as in equation (17) for a GBP yield curve for August 10th, 2000.

deformations, and the higher modes contribute only little due to their lower principal components (or eigenvalues) and therefore lower probability of occurring with sufficient amplitude. Furthermore, the average of the swap rate volatilities associated with the higher modes of deformation was found to be very close to the swap rate volatlity obtainable using today's weights, even when the individual realizations were significantly different. Finally, we have conducted realistic tests on the validity of the approximation and reported the numerical results in detail.

A Derivation of formulæ (16) and (17)

Given the definitions (14) and (15) we have

$$SR = \frac{A_1}{B_1} \ . \tag{22}$$

By virtue of (5), one also obtains

$$\frac{\partial P_{j+1}}{\partial f_i} = -P_{j+1} \frac{\tau_i}{1 + f_i \tau_i} \cdot \mathbf{1}_{\{j \geq i\}} \ . \tag{23}$$

From this we can calculate

$$\frac{\partial SR}{\partial f_i} = \frac{1}{B_1}\frac{\partial A_1}{\partial f_i} - \frac{A_1}{B_1^2}\frac{\partial B_1}{\partial f_i} \tag{24}$$

$$= \frac{1}{B_1}\left[P_{i+1}\tau_i + \sum_{j=i}^{n}\frac{\partial P_{j+1}}{\partial f_i} f_j \tau_j\right] - \frac{A_1}{B_1^2}\sum_{j=i}^{n}\frac{\partial P_{j+1}}{\partial f_i}\tau_j \tag{25}$$

$$= \frac{1}{B_1}\left[P_{i+1}\tau_i - \frac{\tau_i}{1+f_i\tau_i}A_i\right] + \frac{A_1}{B_1^2}\frac{\tau_i}{1+f_i\tau_i}B_i \tag{26}$$

which is identical to equation (16). Since the weights ζ_{jk} define the contribution of the forward rate covariance elements to the swap rate volatility, they must satisfy

$$\left\langle \frac{dSR}{SR}\frac{dSR}{SR}\right\rangle = \sum_{j,k=1}^{n}\zeta_{jk}\left\langle \frac{df_j}{f_j}\frac{df_k}{f_k}\right\rangle \tag{27}$$

$$= \sum_{j,k=1}^{n}\frac{\partial SR}{\partial f_j}\frac{f_j}{SR}\left\langle \frac{df_j}{f_j}\frac{df_k}{f_k}\right\rangle \frac{\partial SR}{\partial f_k}\frac{f_k}{SR}\,. \tag{28}$$

Thus, we have

$$\zeta_{jk} = \frac{\partial SR}{\partial f_j}\frac{f_j}{SR}\cdot\frac{\partial SR}{\partial f_k}\frac{f_k}{SR}\,. \tag{29}$$

This, together with equations (16) and (22) gives us equation (17).

B The indefinite integral of the instantaneous covariance

Given the parametrisation of the instantaneous volatility $\sigma_j(t)$ of the forward rate f_j as in equation (20), and the FRA/FRA correlation (21), the indefinite integral of the covariance becomes

$$\begin{aligned}\int\rho_{ij}(t)\sigma_i(t)\sigma_j(t)dt = {} & \mathrm{e}^{-\beta|t_i-t_j|}\frac{1}{4c^3}\cdot \\ & \cdot\Bigg(4ac^2d\Big[\mathrm{e}^{c(t-t_j)}+\mathrm{e}^{c(t-t_i)}\Big]+4c^3d^2t \\ & -4bcd\mathrm{e}^{c(t-t_i)}\Big[c(t-t_i)-1\Big]-4bcd\mathrm{e}^{c(t-t_j)}\Big[c(t-t_j)-1\Big] \\ & +\mathrm{e}^{c(2t-t_i-t_j)}\Bigg(2a^2c^2+2abc\Big[1+c(t_i+t_j-2t)\Big] \\ & +b^2\Big[1+2c^2(t-t_i)(t-t_j)+c(t_i+t_j-2t)\Big]\Bigg)\Bigg)\quad .\end{aligned}$$

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