# A singular Variance Gamma expansion

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## Abstract

We give an analytical expansion for option prices and Black implied volatilities consistent with the Variance Gamma model [MCC98] based on a singular expansion of the standard gamma density in terms of the Dirac functions and its derivatives.

## 1 Introduction

The Variance Gamma option pricing model [MCC98] can be viewed as a geometric Brownian motion that is time-changed by a Gamma process. Formulated as a martingale, an asset price $S_0$ at time $t = 0$ advances to

$$S_t = S_0 \cdot \mathrm{e}^{\omega t + \theta\gamma_t + \sigma W_{\gamma_t}} \tag{1}$$

at time $t$, with $W_\tau$ being a standard Wiener process to time $\tau$, $\gamma_t$ being a gamma process with

$$\mathsf{E}[\gamma_t] = t \qquad\qquad \mathsf{V}[\gamma_t] = \nu t \tag{2}$$

and

$$\omega = \ln\left(1 - (\theta + \sigma^2/2)\nu\right) / \nu \ . \tag{3}$$

The density of the gamma process is

$$\psi_\gamma(g; t, \nu) = \mathrm{e}^{\frac{t}{\nu}\ln(\frac{g}{\nu}) - \frac{g}{\nu} - \ln\Gamma(\frac{t}{\nu})} \Big/ g \tag{4}$$

and its characteristic function is

$$\phi_\gamma(u) := \int \mathrm{e}^{iug}\psi_\gamma(g; t, \nu)\mathrm{d}g = (1 - iu\nu)^{-\frac{t}{\nu}} \ . \tag{5}$$

Madan, Carr, and Chang [MCC98] give "closed" form solutions for European call and put option prices under this model in terms of the degenerate hypergeometric function of two variables and the modified Bessel function of the second kind. As is well documented, and frequently discussed on internet forums, generic numerical implementations of these special functions tend to be either unstable, or resort to numerically intensive internal schemes. An alternative method for the calculation of option prices is to use the characteristic function of the logarithm $X_t := \theta\gamma_t + \sigma W_{\gamma_t}$ of the asset process given by

$$\phi_{\text{VG}}(u) = \left(1 - i\theta u\nu + \sigma^2 u^2\nu/2\right)^{-\frac{t}{\nu}} \tag{6}$$

together with a Fourier transform of the payoff as suggested by Carr and Madan [CM99]. Alas, this approach is also known to be numerically highly unstable unless one can find an analytical approximation for an optimal shift of the contour integral in the complex domain as described by Lord and Kahl using the example of the Heston model [LK07]. The most viable method for numerical evaluation of European call and put option prices is considered to exploit the fact that, conditional on a given gamma variate $\gamma_t$, the density is lognormal, as mentioned in [MCC98], and to integrate standard Black call or put option prices with a modified variance and forward over the gamma density. This gives us

$$V(S_0, K, \sigma, \theta, \nu, t) = \int \psi_\gamma(g; t, \nu) B(F_g, K, \sigma\sqrt{g})\,\mathrm{d}g \tag{7}$$

for the Variance Gamma option price with

$$B(F, K, \varsigma) = \epsilon F \Phi\left(\epsilon(\tfrac{x}{\varsigma} + \tfrac{\varsigma}{2})\right) - \epsilon K \Phi\left(\epsilon(\tfrac{x}{\varsigma} - \tfrac{\varsigma}{2})\right) \tag{8}$$

$$\epsilon = \pm 1 \text{ for calls/puts} \tag{9}$$

$$x = \ln(F/K) \tag{10}$$

$$F_g = S_0 \mathrm{e}^{\omega t + \tilde{\theta} g} \tag{11}$$

$$\tilde{\theta} = \theta + \sigma^2/2 \ . \tag{12}$$

Whilst this approach is viable, it also requires due diligence and care in its implementation, and needs tens of thousands of Black function evaluations to be robust [Sta05], though this number is likely to be reduced significantly if adaptive quadrature schemes are employed. The crux with all of the known methods is that in many applications the kurtosis parameter $\nu$ of Variance Gamma model is set to be a small number which gives rise to the gamma density to be very peaked, unless one considers very short option expiries that are equal to or even smaller than $\nu$. In addition to this outright practical aspect, it is usually also considered desirable to have an implementation that provides a smooth continuity to the standard Black function (8) as $\nu$ is gradually reduced to zero. This, however, is numerically the hardest part since the gamma density, in this limit, converges to a Dirac distribution centered at $t$.

In this note, we give an alternative view on the European option pricing problem with the Variance

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Gamma model that has its own advantages and shortcomings, as we shall see. It is based on a singular expansion of the gamma density which is worth mentioning in its own right as it can be applied in any context.

## 2 Singular expansion of the gamma density

We begin by noting that the gamma density permits no regular expansion for small values of $\nu$ since, as mentioned, it converges to a Dirac function for $\nu \to 0$. In contrast, however, its characteristic function is well behaved for small $\nu$ as it allows the Taylor expansion in $\nu$

$$\phi_\gamma(u) = \mathrm{e}^{iut}\left[1 - u^2\tfrac{t}{2}\nu + (u^4\tfrac{t^2}{8} - iu^3\tfrac{t}{3})\nu^2 + \ldots\right] . \quad (13)$$

Our next observation is

$$\mathrm{e}^{iut}(iu)^k = \mathrm{d}_t^k \mathrm{e}^{iut} = \int \mathrm{e}^{iug}(-1)^k\delta^{(k)}(g-t)\mathrm{d}g \quad (14)$$

where $\delta^{(k)}(\cdot)$ represents the $k$-th derivative of the Dirac function. By identification of terms, we therefore arrive at an expansion of the gamma density in $\nu$ in terms of the Dirac function and its derivatives:

$$\begin{aligned}\psi_\gamma(g;t,\nu) &= \delta(g-t) + \nu\cdot\tfrac{t}{2}\delta^{(2)}(g-t) \quad (15)\\ &\quad + \nu^2\cdot\left(\tfrac{t^2}{8}\delta^{(4)}(g-t) - \tfrac{t}{3}\delta^{(3)}(g-t)\right) + \mathcal{O}(\nu^3)\end{aligned}$$

As is the nature with all functions that involve Dirac terms, this expansion does of course make no sense if we needed to evaluate it directly by virtue of the fact that it consists of series of exclusively *singular* contributions. However, any integral over the gamma density with this expansion results in a perfectly sensible series in terms of the integrand and its derivatives:

$$\begin{aligned}\int\psi_\gamma(g;t,\nu)f(g)\mathrm{d}g &= f(t) + \nu\tfrac{t}{2}f''(t) \quad (16)\\ &\quad + \nu^2\left(\tfrac{t}{3}f'''(t) + \tfrac{t^2}{8}f^{(4)}(t)\right) + \mathcal{O}(\nu^3) .\end{aligned}$$

**Remark**. By the same token, we can derive for the Gauss density

$$\begin{aligned}\varphi(z;\sigma) &= \mathrm{e}^{-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2}\Big/(\sigma\sqrt{2\pi}) \quad (17)\\ &= \delta(z) + \tfrac{\sigma^2}{2}\delta''(z) + \tfrac{\sigma^4}{8}\delta^{(4)}(z) + \tfrac{\sigma^6}{48}\delta^{(6)}(z) + \mathcal{O}(\sigma^8) \quad (18)\end{aligned}$$

which is closely related, or arguably equivalent, to the idea of saddle-point approximations.

## 3 Singular Variance Gamma expansion

We now apply the expansion (15) to the Variance Gamma option price calculation. First, we define the normalized Black option value function as the ratio of the price and the geometric average of forward and strike:

$$\begin{aligned}b(x,\varsigma) &:= B(F,K,\varsigma)/\sqrt{FK} \quad (19)\\ &= \epsilon\mathrm{e}^{\frac{x}{2}}\Phi\left(\epsilon(\tfrac{x}{\varsigma}+\tfrac{\varsigma}{2})\right) - \epsilon\mathrm{e}^{-\frac{x}{2}}\Phi\left(\epsilon(\tfrac{x}{\varsigma}-\tfrac{\varsigma}{2})\right) .\end{aligned}$$

In analogy, we define for the Variance Gamma option value:

$$\begin{aligned}v(x,\sigma,\theta,\nu,t) &:= V(F,K,\sigma,\theta,\nu,t)/\sqrt{FK} \quad (20)\\ &= \int\psi_\gamma(g;t,\nu)\mathrm{e}^{\frac{1}{2}(\omega t+\tilde\theta g)}b(x+\omega t+\tilde\theta g,\sigma\sqrt{g})\,\mathrm{d}g .\end{aligned}$$

Application of equation (16) to first order in $\nu$, after some algebraic simplification, yields

$$\begin{aligned}&v(x,\sigma,\theta,\nu,t) = \quad (21)\\ &b + \left[\tfrac{x^2}{4\varsigma} - (\tfrac{x}{\varsigma}-\tfrac{\varsigma}{2})\tilde\theta t - \tfrac{\varsigma}{4}(\tfrac{\varsigma^2}{4}+1) + \tfrac{\tilde\theta^2t^2}{\varsigma}\right]\tfrac{\partial_\varsigma b}{2t}\nu + \mathcal{O}(\nu^2)\end{aligned}$$

with $b = b(x,\varsigma)|_{\varsigma=\sigma\sqrt{t}}$ and

$$\partial_\varsigma b = \partial_\varsigma b(x,\varsigma)\Big|_{\varsigma=\sigma\sqrt{t}} = \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{\varsigma^2}+\frac{\varsigma^2}{4}\right)}}{\sqrt{2\pi}}\Bigg|_{\varsigma=\sigma\sqrt{t}} . \quad (22)$$

Equation (21) represents the Variance Gamma option price as the Black option price plus a correction term of first order in $\nu$. We can translate this into a Black implied volatility approximation

$$\hat\sigma_{\mathrm{VG}} := \hat\sigma_0 + \hat\sigma_1\nu + \mathcal{O}(\nu^2) \quad (23)$$

for the Variance Gamma model by equating (21) with

$$b(x,\varsigma)\Big|_{\varsigma=\hat\sigma_{\mathrm{VG}}\sqrt{t}} = b(x,\varsigma)\Big|_{\varsigma=\hat\sigma_0\sqrt{t}} + \partial_\varsigma b(x,\varsigma)\Big|_{\varsigma=\hat\sigma_0\sqrt{t}}\hat\sigma_1\sqrt{t}\cdot\nu + \mathcal{O}(\nu^2) \quad (24)$$

to obtain

$$\begin{aligned}\hat\sigma_0 &= \sigma \quad (25)\\ \hat\sigma_1 &= \tfrac{x^2}{8\sigma t^2} - \tfrac{\sigma}{8}(\tfrac{\sigma^2}{4}+\tfrac{1}{t}) + (\tfrac{\sigma}{4}-\tfrac{x}{2\sigma t})\tilde\theta + \tfrac{\tilde\theta^2}{2\sigma} . \quad (26)\end{aligned}$$

The simplicity of this procedure makes it easy to utilize the power of modern computer algebra systems to derive higher order expansions for $\hat\sigma_{\mathrm{VG}}$. In appendix A, we give the approximation up to order $\mathcal{O}(\nu^5)$ derived using the open source software Maxima [Max]. The Maxima code [Jäc09] used for our calculations is straightforward and should be easy to translate to other computer algebra systems.

Of particular interest in analytical approximations for implied volatilities for any model are the at-the-money volatility

$$\begin{aligned}\hat\sigma_{\mathrm{VG}}\Big|_{K=S_0} &= \sigma + \left(-\tfrac{\sigma^3}{32} + \tfrac{\sigma\tilde\theta}{4} + \tfrac{\tilde\theta^2}{2\sigma} - \tfrac{\sigma}{8t}\right)\nu \quad (27)\\ &\quad + \left(\tfrac{13\sigma^5}{6144} - \tfrac{3\sigma^3\tilde\theta}{128} + \tfrac{9\sigma\tilde\theta^2}{64} + \tfrac{3\tilde\theta^3}{8\sigma} - \tfrac{\tilde\theta^4}{8\sigma^3}\right.\\ &\quad\left. + \tfrac{\sigma}{128t^2} + \tfrac{\sigma^3}{192t} - \tfrac{\sigma\tilde\theta}{32t} + \tfrac{\tilde\theta^2}{16\sigma t}\right)\nu^2 + \mathcal{O}(\nu^3) ,\end{aligned}$$

the (proportional) at-the-money skew

$$\frac{1}{K}\mathrm{d}_K\hat\sigma_{\mathrm{VG}}\bigg|_{K=S_0} = -\,\mathrm{d}_x\hat\sigma_{\mathrm{VG}}\bigg|_{K=S_0} \tag{28}$$
$$= \tfrac{\tilde\theta}{2\sigma t}\nu\left[1+\left(\tfrac{\sigma^2}{32}+\tfrac{3}{8t}+\tfrac{\tilde\theta}{4}-\tfrac{5\tilde\theta^2}{6\sigma^2}\right)\nu\right]+\mathcal{O}(\nu^3)\,,$$

and the (proportional) at-the-money smile curvature

$$\frac{1}{K^2}\mathrm{d}^2_K\hat\sigma_{\mathrm{VG}}\bigg|_{K=S_0} = \mathrm{d}_x\hat\sigma_{\mathrm{VG}}\bigg|_{K=S_0} + \mathrm{d}^2_x\hat\sigma_{\mathrm{VG}}\bigg|_{K=S_0} \tag{29}$$
$$= \left(\tfrac{1}{4\sigma t^2}-\tfrac{\tilde\theta}{2\sigma t}\right)\nu + \left(\tfrac{3}{16\sigma t^3}+\tfrac{5\sigma}{384t^2}-\tfrac{\tilde\theta}{4\sigma t^2}-\tfrac{9\tilde\theta^2}{8\sigma^3t^2} -\tfrac{\sigma\tilde\theta}{64t}-\tfrac{\tilde\theta^2}{8\sigma t}+\tfrac{5\tilde\theta^3}{12\sigma^3 t}\right)\nu^2+\mathcal{O}(\nu^3)\,.$$

We can see that, to first order, the skew is given by $\frac{1}{2}\frac{\tilde\theta\nu}{\sigma t}$ and thus decays like $1/t$. For $\tilde\theta=0$, the at-the-money smile curvature is $\frac{1}{4}\frac{\nu}{\sigma t^2}$ which means that the smile decays like $1/t^2$.

### 3.1 Time-dependent parameters

In practice, one may wish to use the Variance Gamma model with time-dependent parameters $\sigma(t)$, $\theta(t)$ and $\nu(t)$ for greater flexibility. Starting with the simplified two-epoch setting in which the Variance Gamma process has constant parameters $\sigma_1$, $\theta_1$, and $\nu_1$ between $t=0$ and $t=\tau_1$, and constant parameters $\sigma_2$, $\theta_2$, and $\nu_2$ between $t=\tau_1$ and $t=\tau_1+\tau_2$, it is tempting to base a time dependency approximation on the Taylor expansion of (5) in $\nu$:

$$\phi_{\mathrm{VG}}(u) = \mathrm{e}^{(iu\theta-u^2\frac{\sigma^2}{2})t} \cdot \left[1+\left(\tfrac{1}{8}\sigma^4u^4-i\tfrac{1}{2}\sigma^2\theta u^3-\tfrac{1}{2}\theta^2u^2\right)t\nu+\mathcal{O}(\nu^2)\right]\,. \tag{30}$$

Expanding the product of the characteristic functions for the transition densities over the two time steps, and matching it with the characteristic function for a single step transition density from $t=0$ to $t=\tau_1+\tau_2$, one may wish to attempt using

$$\bar\sigma = \sqrt{(\sigma_1^2\tau_1+\sigma_2^2\tau_2)/(\tau_1+\tau_2)} \tag{31}$$
$$\bar\theta = (\theta_1\tau_1+\theta_2\tau_2)/(\tau_1+\tau_2) \tag{32}$$
$$\bar\nu = \frac{\nu_1\sigma_1^4\tau_1+\nu_2\sigma_2^4\tau_2}{\bar\sigma^4\cdot(\tau_1+\tau_2)} \tag{33}$$

in order to match the exponential terms and highest power in $u$ within first order in $\nu$. If we extend this to multiple time steps with different parameters levels, and ultimately take it to the limit of infinitely many infinitesimal time steps, we arrive at the first order parameter averaging rule

$$\bar\sigma(T) = \sqrt{\int_0^T\sigma^2(t)\mathrm{d}t\Big/T} \tag{34}$$
$$\bar\theta(T) = \int_0^T\theta(t)\mathrm{d}t\Big/T \tag{35}$$
$$\bar\nu(T) = \frac{\int_0^T\sigma^4(t)\nu(t)\mathrm{d}t}{\bar\sigma^4(T)\cdot T} \tag{36}$$

for arbitrarily time-dependent (instaneous) process parameters $\sigma(t)$, $\theta(t)$, and $\nu(t)$. In practice, however, we found that this approximation only works for mildly varying parameters.

An alternative approach is to repeat the singular expansion in two gamma variates. This gives, after considerable algebraic simplification, to first order in both $\nu_1$ and $\nu_2$,

$$\hat\sigma_{\mathrm{VG}} \approx \hat\sigma_0+\tfrac{\sigma_1^4\tau_1\nu_1+\sigma_2^4\tau_2\nu_2}{\hat\sigma_0^4T}\left(\tfrac{x^2}{8\hat\sigma_0T^2}-\tfrac{\hat\sigma_0}{8}\left(\tfrac{\hat\sigma_0^2}{4}+\tfrac{1}{T}\right)\right) + \tfrac{\sigma_1^2\tau_1\tilde\theta_1\nu_1+\sigma_2^2\tau_2\tilde\theta_2\nu_2}{\hat\sigma_0^2T}\left(\tfrac{\hat\sigma_0}{4}-\tfrac{x}{2\hat\sigma_0T}\right)+\tfrac{\tilde\theta_1^2\tau_1\nu_1+\tilde\theta_2^2\tau_2\nu_2}{2\hat\sigma_0T}\,, \tag{37}$$

with

$$\hat\sigma_0 = \sqrt{(\sigma_1^2\tau_1+\sigma_2^2\tau_2)/(\tau_1+\tau_2)}\,, \tag{38}$$
$$\tilde\theta_i = \theta_i+\sigma_i^2/2 \tag{39}$$
$$T = \tau_1+\tau_2\,. \tag{40}$$

From here, it is a matter of exceptionally tedious but unsurprising calculations to prove that this generalizes to

$$\hat\sigma_{\mathrm{VG}} \approx \hat\sigma_0+\tfrac{\int_0^T\sigma^4(t)\nu(t)\mathrm{d}t}{\hat\sigma_0^4\cdot T}\left(\tfrac{x^2}{8\hat\sigma_0T^2}-\tfrac{\hat\sigma_0}{8}\left(\tfrac{\hat\sigma_0^2}{4}+\tfrac{1}{T}\right)\right) + \tfrac{\int_0^T\sigma^4(t)\tilde\theta(t)\nu(t)\mathrm{d}t}{\hat\sigma_0^2\cdot T}\left(\tfrac{\hat\sigma_0}{4}-\tfrac{x}{2\hat\sigma_0T}\right)+\tfrac{\int_0^T\tilde\theta(t)\nu(t)\mathrm{d}t}{2\hat\sigma_0T}\,, \tag{41}$$

with

$$\hat\sigma_0 = \sqrt{\int_0^T\sigma^2(t)\mathrm{d}t\Big/T}\,, \tag{42}$$
$$\tilde\theta(t) = \theta(t)+\sigma^2(t)/2\,, \tag{43}$$

to first order in $\nu(t)$ for the arbitrarily time-dependent case.

## 4 Numerical examples

In figures 1 to 6, we can see the quality of the analytical approximations of orders up to $\mathcal{O}(\nu^5)$ for $\sigma=25\%$, $\theta=-1/4$, $\nu=1/10$, and $t=10,5,1,\frac{1}{2},\frac{1}{4},\frac{1}{10}$. In figure 8, we show the quality of the analytical approximation for the at-the-money volatility for the same parameters as in figures 1 to 6, and similarly we show the at-the-money skew and at-the-money smile curvature in figures 9 and 10. It is clear that the strike range of validity shrinks as $t$ decreases, and that for $t\lesssim\nu$, the approximation breaks down.

Finally, we show in figures 11 and 12 two examples for the quality of the time-average parameter approximations (31) to (33) and the time-dependent parameter expansion (37).

## 5 Conclusion

We presented an analytical approximation for Black implied volatilies generated by the Variance Gamma

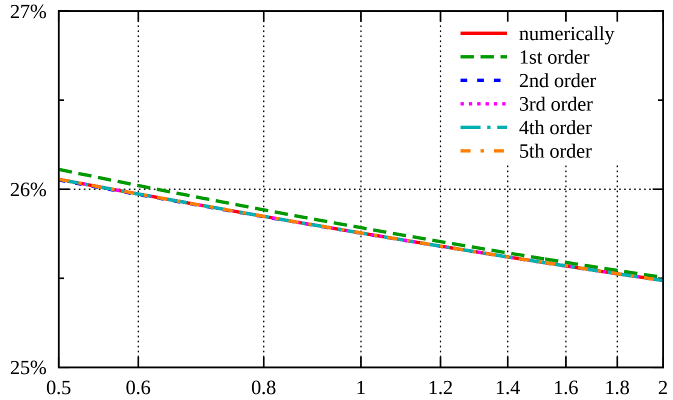


Figure 1: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 10$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

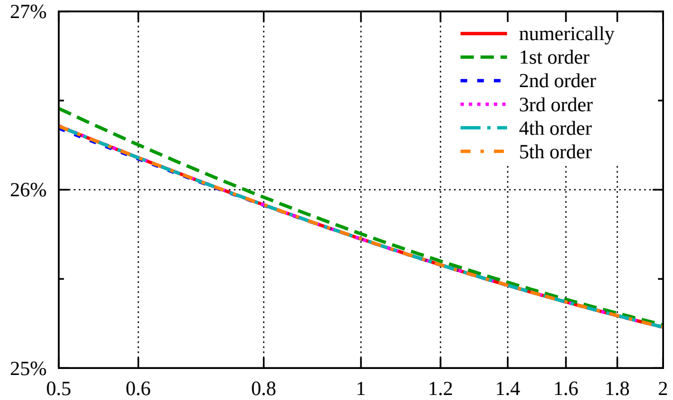


Figure 2: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 5$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

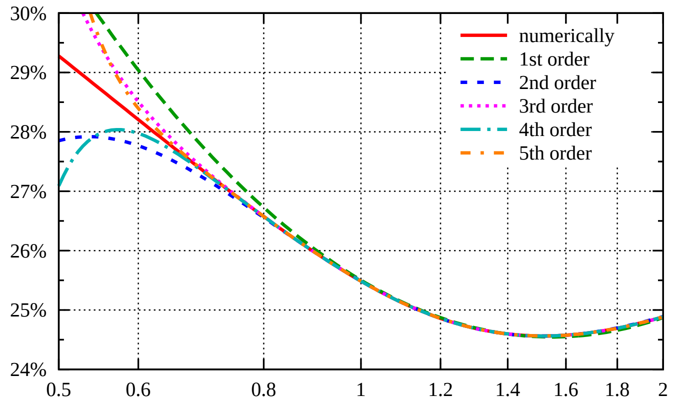


Figure 3: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 1$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

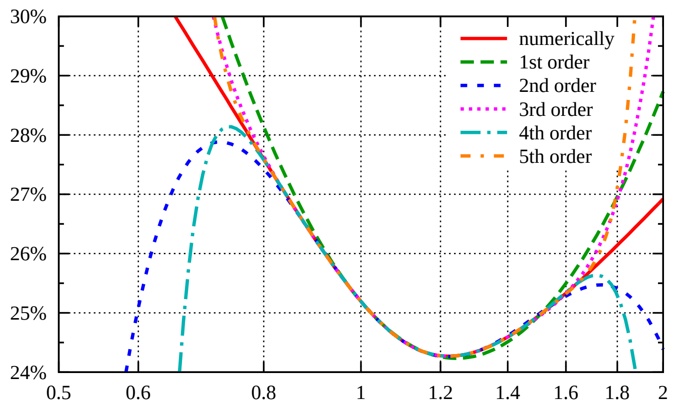


Figure 4: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 1/2$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

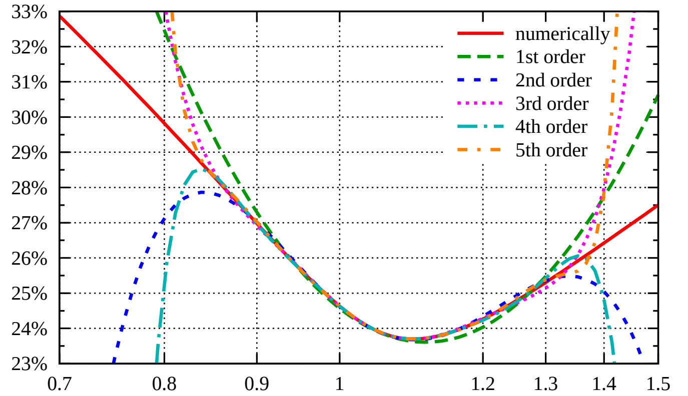


Figure 5: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 1/4$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

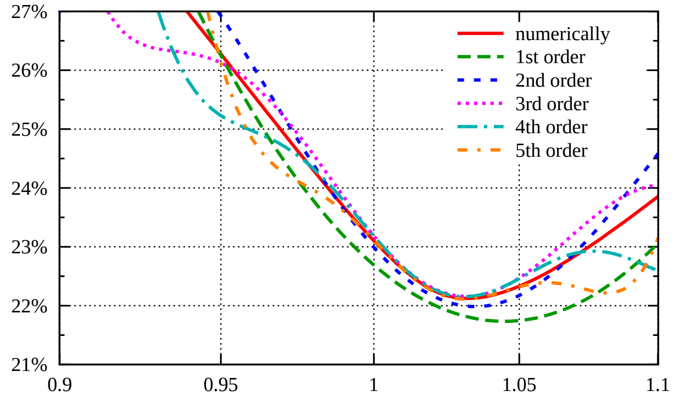


Figure 6: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 1/10$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

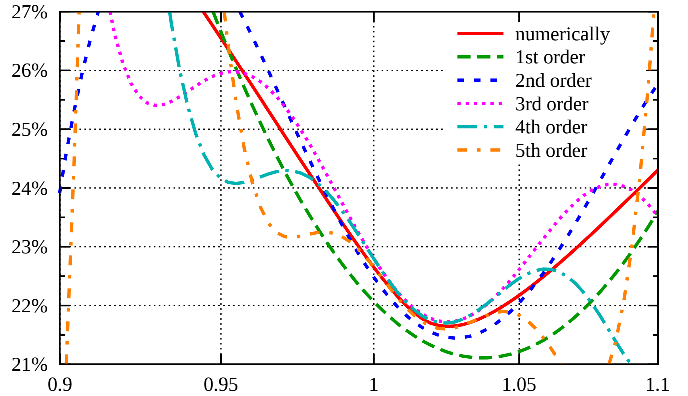


Figure 7: Implied volatilities as a function of $K/S_0$ for Variance Gamma model with $t = 1/12$, $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

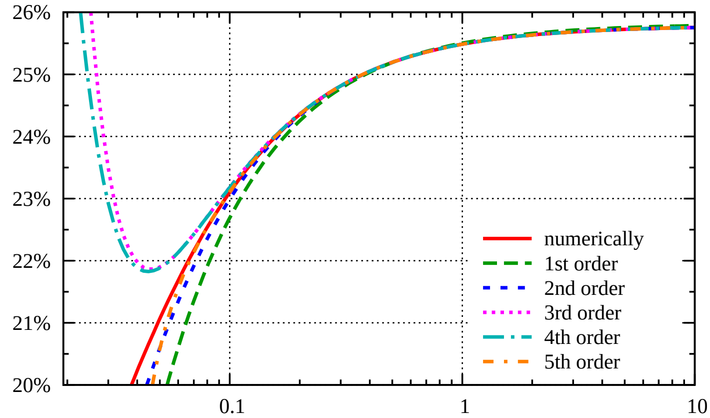


Figure 8: At-the-money implied volatility $\hat{\sigma}_{\mathrm{VG}}|_{K=S_0}$ as a function of $t$ for $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

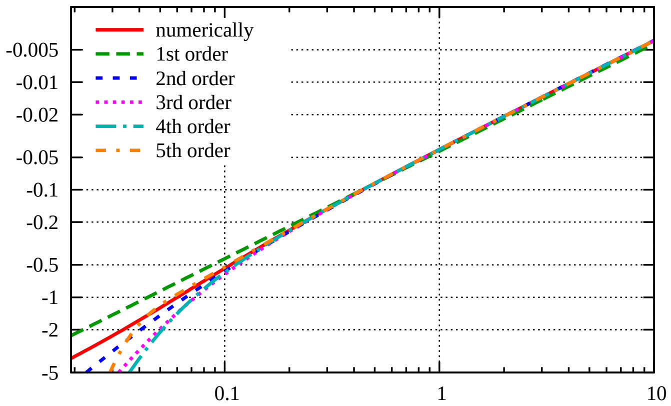


Figure 9: At-the-money implied volatility skew $\frac{1}{S_0}\mathrm{d}_K\hat{\sigma}_{\mathrm{VG}}|_{K=S_0}$ as a function of $t$ for $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

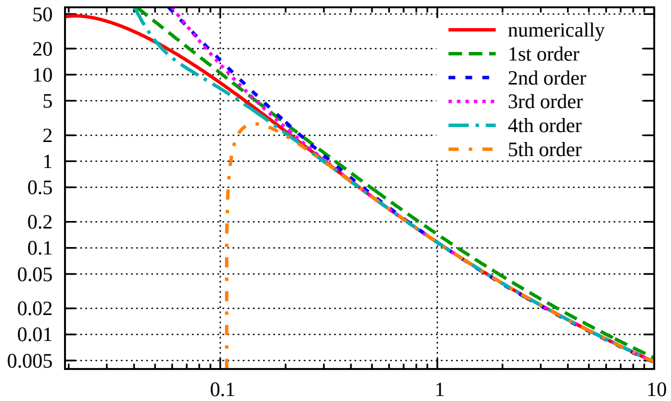


Figure 10: At-the-money smile curvature $\frac{1}{S_0^2}\mathrm{d}_K^2\hat{\sigma}_{\mathrm{VG}}|_{K=S_0}$ as a function of $t$ for $\sigma = 25\%$, $\theta = -1/4$, $\nu = 1/10$.

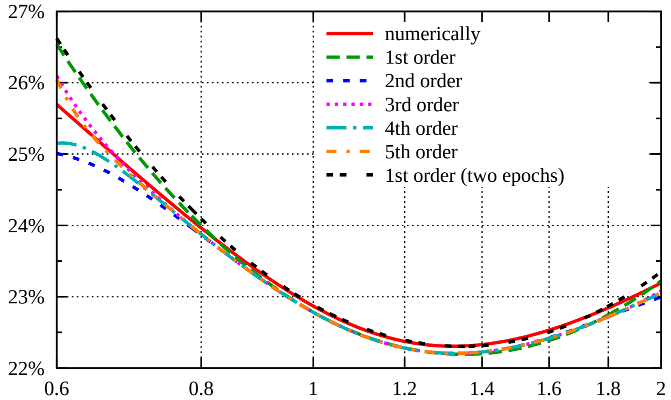


Figure 11: Implied volatilities for piecewise constant parameters with $\sigma_1 = 25\%$, $\theta_1 = -0.15$, $\nu_1 = 0.1$, $\tau_1 = 0.5$, $\sigma_2 = 20\%$, $\theta_2 = -0.2$, $\nu_2 = 0.15$, and $\tau_2 = 0.5$. The curves labelled "1st order" to "5th order" are generated using the constant parameter expansion with the time-averaged parameters given in (31) to (33). The curve labelled "1st order (two epochs)" corresponds to expansion (37).

model [MCC98] based on a singular expansion of the gamma density. The approach is in spirit not too dissimilar to saddle-point approximations but requires no numerical calculations whatsoever, not even a one-dimensional root finding as would be needed for conventional saddle-point calculations. The mechanistic nature of the used methodology makes it readily

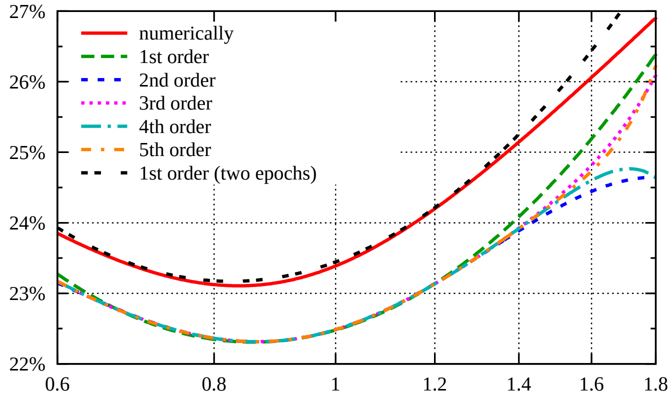


Figure 12: Implied volatilities for piecewise constant parameters with $\sigma_1 = 25\%$, $\theta_1 = -0.1$, $\nu_1 = 0.1$, $\tau_1 = 0.5$, $\sigma_2 = 20\%$, $\theta_2 = 0.2$, $\nu_2 = 0.2$, and $\tau_2 = 0.5$. The curves labelled "1st order" to "5th order" are generated using the constant parameter expansion with the time-averaged parameters given in (31) to (33). The curve labelled "1st order (two epochs)" corresponds to expansion (37).

amenable to the use of computer algebra systems. We used the open source system Maxima for our computations, though, the code could easily be translated to any other symbolic mathematics package.

The results we obtained give an insight into the asymptotic behaviour for $\frac{\nu}{t} \ll 1$, i.e., for $\nu$ small or $t$ sizeable. In particular, we obtain accurate explicit results for the at-the-money volatility, the skew, and the curvature of the implied volatility smile generated by the Variance Gamma model. For short maturities $t < \nu$, the quality of the results breaks down due to the fact that the gamma density loses its uni-modal shape that is the basis of its expansion in terms of the Dirac distribution and its derivatives.

It remains to be seen to what extent the presented results can be of practical use beyond giving an analytically precise understanding of the asymptotic behaviour of the Variance Gamma model for small $\nu/t$.

# A Fifth order expansion for $\hat{\sigma}_{\mathrm{VG}}$

The Black implied volatility for the Variance Gamma option price is given by

$$\hat{\sigma}_{\mathrm{VG}} \approx \hat{\sigma}_0 + \hat{\sigma}_1\nu + \hat{\sigma}_2\nu^2 + \hat{\sigma}_3\nu^3 + \hat{\sigma}_4\nu^4 + \hat{\sigma}_5\nu^5 \tag{44}$$

with

$$\hat{\sigma}_0 = \sigma \tag{45}$$

$$\hat{\sigma}_1 = \frac{x^2}{8\sigma t^2} - \frac{\sigma}{8}\left(\frac{\sigma^2}{4} + \frac{1}{t}\right) + \left(\frac{\sigma}{4} - \frac{x}{2\sigma t}\right)\tilde{\theta} + \frac{\tilde{\theta}^2}{2\sigma} \tag{46}$$

$$\begin{aligned}\hat{\sigma}_2 = {} & \frac{13\sigma^5}{6144} - \frac{3\sigma^3\tilde{\theta}}{128} + \frac{9\sigma\tilde{\theta}^2}{64} + \frac{3\tilde{\theta}^3}{8\sigma} - \frac{\tilde{\theta}^4}{8\sigma^3} + \frac{\sigma}{128t^2} + \frac{\sigma^3}{192t} - \frac{\sigma\tilde{\theta}}{32t} + \\ & \frac{\tilde{\theta}^2}{16\sigma t} - \frac{3\tilde{\theta}x}{16\sigma t^2} - \frac{\sigma\tilde{\theta}x}{64t} - \frac{\tilde{\theta}^2 x}{8\sigma t} + \frac{5\tilde{\theta}^3 x}{12\sigma^3 t} + \frac{3x^2}{32\sigma t^3} + \frac{5\sigma x^2}{768t^2} - \\ & \frac{\tilde{\theta}x^2}{32\sigma t^2} - \frac{9\tilde{\theta}^2 x^2}{16\sigma^3 t^2} + \frac{5\tilde{\theta}x^3}{16\sigma^3 t^3} - \frac{23x^4}{384\sigma^3 t^4}\end{aligned} \tag{47}$$

$$
\begin{aligned}
\hat{\sigma}_3 = &-\frac{35\sigma^7}{196608} + \frac{65\sigma^5\tilde{\theta}}{24576} - \frac{223\sigma^3\tilde{\theta}^2}{12288} + \frac{25\sigma\tilde{\theta}^3}{256} + \frac{215\tilde{\theta}^4}{768\sigma} - \frac{5\tilde{\theta}^5}{32\sigma^3} + \\
&\frac{\tilde{\theta}^6}{16\sigma^5} + \frac{5\sigma}{1024t^3} - \frac{\sigma^3}{12288t^2} + \frac{\sigma\tilde{\theta}}{512t^2} + \frac{13\tilde{\theta}^2}{256\sigma t^2} - \frac{19\sigma^5}{49152t} + \frac{\sigma^3\tilde{\theta}}{256t} - \\
&\frac{3\sigma\tilde{\theta}^2}{256t} + \frac{3\tilde{\theta}^3}{64\sigma t} - \frac{11\tilde{\theta}^4}{64\sigma^3 t} - \frac{25\tilde{\theta}x}{256\sigma t^3} - \frac{\sigma\tilde{\theta}x}{128t^2} - \frac{3\tilde{\theta}^2x}{64\sigma t^2} + \frac{23\tilde{\theta}^3x}{32\sigma^3t^2} + \\
&\frac{7\sigma^3\tilde{\theta}x}{12288t} - \frac{3\sigma\tilde{\theta}^2x}{256t} - \frac{5\tilde{\theta}^3x}{384\sigma t} + \frac{5\tilde{\theta}^4x}{16\sigma^3t} - \frac{7\tilde{\theta}^5x}{16\sigma^5t} + \frac{55x^2}{1024\sigma t^4} + \frac{9\sigma x^2}{2048t^3} - \\
&\frac{3\tilde{\theta}x^2}{128\sigma t^3} - \frac{17\tilde{\theta}^2x^2}{16\sigma^3t^3} - \frac{11\sigma^3x^2}{49152t^2} + \frac{5\sigma\tilde{\theta}x^2}{3072t^2} - \frac{127\tilde{\theta}^2x^2}{1536\sigma t^2} - \frac{9\tilde{\theta}^3x^2}{64\sigma^3t^2} + \\
&\frac{75\tilde{\theta}^4x^2}{64\sigma^5t^2} + \frac{43\tilde{\theta}x^3}{64\sigma^3t^4} + \frac{67\tilde{\theta}x^3}{1536\sigma t^3} - \frac{5\tilde{\theta}^2x^3}{64\sigma^3t^3} - \frac{49\tilde{\theta}^3x^3}{32\sigma^5t^3} - \frac{467x^4}{3072\sigma^3t^5} - \\
&\frac{113x^4}{12288\sigma t^4} + \frac{23\tilde{\theta}x^4}{512\sigma^3t^4} + \frac{271\tilde{\theta}^2x^4}{256\sigma^5t^4} - \frac{95\tilde{\theta}x^5}{256\sigma^5t^5} + \frac{53x^6}{1024\sigma^5t^6}
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\hat{\sigma}_4 = &\frac{6271\sigma^9}{377487360} - \frac{245\sigma^7\tilde{\theta}}{786432} + \frac{1055\sigma^5\tilde{\theta}^2}{393216} - \frac{721\sigma^3\tilde{\theta}^3}{49152} + \frac{3667\sigma\tilde{\theta}^4}{49152} + \\
&\frac{665\tilde{\theta}^5}{3072\sigma} - \frac{683\tilde{\theta}^6}{4608\sigma^3} + \frac{7\tilde{\theta}^7}{64\sigma^5} - \frac{5\tilde{\theta}^8}{128\sigma^7} - \frac{21\sigma}{32768t^4} - \frac{7\sigma^3}{24576t^3} + \\
&\frac{5\sigma\tilde{\theta}}{4096t^3} + \frac{55\tilde{\theta}^2}{2048\sigma t^3} - \frac{17\sigma^5}{3932160t^2} - \frac{\sigma^3\tilde{\theta}}{16384t^2} + \frac{27\sigma\tilde{\theta}^2}{8192t^2} + \frac{39\tilde{\theta}^3}{1024\sigma t^2} - \\
&\frac{289\tilde{\theta}^4}{1024\sigma^3t^2} + \frac{403\sigma^7}{11796480t} - \frac{95\sigma^5\tilde{\theta}}{196608t} + \frac{275\sigma^3\tilde{\theta}^2}{98304t} - \frac{5\sigma\tilde{\theta}^3}{1024t} + \frac{3\tilde{\theta}^4}{256\sigma t} - \\
&\frac{55\tilde{\theta}^5}{256\sigma^3t} + \frac{103\tilde{\theta}^6}{384\sigma^5t} - \frac{105\tilde{\theta}x}{2048\sigma t^4} - \frac{35\sigma\tilde{\theta}x}{8192t^3} - \frac{25\tilde{\theta}^2x}{1024\sigma t^3} + \frac{1745\tilde{\theta}^3x}{1536\sigma^3t^3} + \\
&\frac{9\sigma^3\tilde{\theta}x}{32768t^2} - \frac{3\sigma\tilde{\theta}^2x}{512t^2} + \frac{121\tilde{\theta}^3x}{1536\sigma t^2} + \frac{69\tilde{\theta}^4x}{128\sigma^3t^2} - \frac{1141\tilde{\theta}^5x}{640\sigma^5t^2} - \frac{5\sigma^5\tilde{\theta}x}{131072t} + \\
&\frac{35\sigma^3\tilde{\theta}^2x}{49152t} - \frac{623\sigma\tilde{\theta}^3x}{73728t} + \frac{35\tilde{\theta}^4x}{1536\sigma t} + \frac{377\tilde{\theta}^5x}{2560\sigma^3t} - \frac{35\tilde{\theta}^6x}{64\sigma^5t} + \frac{15\tilde{\theta}^7x}{32\sigma^7t} + \\
&\frac{105x^2}{4096\sigma t^5} + \frac{203\sigma x^2}{98304t^4} - \frac{55\tilde{\theta}x^2}{4096\sigma t^4} - \frac{3479\tilde{\theta}^2x^2}{2048\sigma^3t^4} - \frac{251\sigma^3x^2}{1474560t^3} + \\
&\frac{9\sigma\tilde{\theta}x^2}{8192t^3} - \frac{1979\tilde{\theta}^2x^2}{12288\sigma t^3} - \frac{17\tilde{\theta}^3x^2}{64\sigma^3t^3} + \frac{1195\tilde{\theta}^4x^2}{256\sigma^5t^3} + \frac{113\sigma^5x^2}{7864320t^2} - \\
&\frac{11\sigma^3\tilde{\theta}x^2}{65536t^2} - \frac{3\sigma\tilde{\theta}^2x^2}{32768t^2} - \frac{123\tilde{\theta}^3x^2}{2048\sigma t^2} + \frac{1171\tilde{\theta}^4x^2}{6144\sigma^3t^2} + \frac{225\tilde{\theta}^5x^2}{256\sigma^5t^2} - \\
&\frac{767\tilde{\theta}^6x^2}{384\sigma^7t^2} + \frac{2283\tilde{\theta}x^3}{2048\sigma^3t^5} + \frac{1225\tilde{\theta}x^3}{12288\sigma t^4} - \frac{43\tilde{\theta}^2x^3}{256\sigma^3t^4} - \frac{601\tilde{\theta}^3x^3}{96\sigma^5t^4} + \\
&\frac{61\sigma\tilde{\theta}x^3}{98304t^3} + \frac{67\tilde{\theta}^2x^3}{6144\sigma t^3} - \frac{3479\tilde{\theta}^3x^3}{9216\sigma^3t^3} - \frac{49\tilde{\theta}^4x^3}{128\sigma^5t^3} + \frac{2787\tilde{\theta}^5x^3}{640\sigma^7t^3} - \\
&\frac{4427x^4}{16384\sigma^3t^6} - \frac{1163x^4}{49152\sigma t^5} + \frac{467\tilde{\theta}x^4}{4096\sigma^3t^5} + \frac{9345\tilde{\theta}^2x^4}{2048\sigma^5t^5} - \frac{223\sigma x^4}{1310720t^4} + \\
&\frac{113\tilde{\theta}x^4}{49152\sigma t^4} + \frac{2087\tilde{\theta}^2x^4}{8192\sigma^3t^4} - \frac{271\tilde{\theta}^3x^4}{1024\sigma^5t^4} - \frac{5683\tilde{\theta}^4x^4}{1024\sigma^7t^4} - \frac{3519\tilde{\theta}x^5}{2048\sigma^5t^6} - \\
&\frac{2269\tilde{\theta}x^5}{24576\sigma^3t^5} + \frac{285\tilde{\theta}^2x^5}{1024\sigma^5t^5} + \frac{6623\tilde{\theta}^3x^5}{1536\sigma^7t^5} + \frac{1003x^6}{3840\sigma^5t^7} + \frac{19919x^6}{1474560\sigma^3t^6} - \\
&\frac{265\tilde{\theta}x^6}{4096\sigma^5t^6} - \frac{4121\tilde{\theta}^2x^6}{2048\sigma^7t^6} + \frac{1061\tilde{\theta}x^7}{2048\sigma^7t^7} - \frac{27763x^8}{491520\sigma^7t^8}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\hat{\sigma}_5 = &-\frac{2211\sigma^{11}}{1342177280} + \frac{6271\sigma^9\tilde{\theta}}{167772160} - \frac{32929\sigma^7\tilde{\theta}^2}{83886080} + \frac{1345\sigma^5\tilde{\theta}^3}{524288} - \\
&\frac{57803\sigma^3\tilde{\theta}^4}{4718592} + \frac{3945\sigma\tilde{\theta}^5}{65536} + \frac{256031\tilde{\theta}^6}{1474560\sigma} - \frac{263\tilde{\theta}^7}{2048\sigma^3} + \frac{1559\tilde{\theta}^8}{12288\sigma^5} - \\
&\frac{45\tilde{\theta}^9}{512\sigma^7} + \frac{7\tilde{\theta}^{10}}{256\sigma^9} - \frac{399\sigma}{262144t^5} - \frac{79\sigma^3}{1048576t^4} - \frac{21\sigma\tilde{\theta}}{131072t^4} + \\
&\frac{819\tilde{\theta}^2}{65536\sigma t^4} + \frac{649\sigma^5}{31457280t^3} - \frac{7\sigma^3\tilde{\theta}}{32768t^3} + \frac{51\sigma\tilde{\theta}^2}{32768t^3} + \frac{165\tilde{\theta}^3}{8192\sigma t^3} - \\
&\frac{3545\tilde{\theta}^4}{8192\sigma^3t^3} + \frac{367\sigma^7}{377487360t^2} - \frac{17\sigma^5\tilde{\theta}}{3145728t^2} - \frac{221\sigma^3\tilde{\theta}^2}{1572864t^2} + \\
&\frac{115\sigma\tilde{\theta}^3}{32768t^2} - \frac{1291\tilde{\theta}^4}{98304\sigma t^2} - \frac{1445\tilde{\theta}^5}{4096\sigma^3t^2} + \frac{5431\tilde{\theta}^6}{6144\sigma^5t^2} - \frac{3313\sigma^9}{1006632960t} + \\
&\frac{2821\sigma^7\tilde{\theta}}{47185920t} - \frac{2801\sigma^5\tilde{\theta}^2}{5898240t} + \frac{805\sigma^3\tilde{\theta}^3}{393216t} - \frac{291\sigma\tilde{\theta}^4}{131072t} - \frac{7\tilde{\theta}^5}{512\sigma t} - \\
&\frac{943\tilde{\theta}^6}{6144\sigma^3t} + \frac{721\tilde{\theta}^7}{1536\sigma^5t} - \frac{1145\tilde{\theta}^8}{3072\sigma^7t} - \frac{1659\tilde{\theta}x}{65536\sigma t^5} - \frac{33\sigma\tilde{\theta}x}{16384t^4} - \\
&\frac{105\tilde{\theta}^2x}{8192\sigma t^4} + \frac{7105\tilde{\theta}^3x}{4096\sigma^3t^4} + \frac{257\sigma^3\tilde{\theta}x}{1572864t^3} - \frac{105\sigma\tilde{\theta}^2x}{32768t^3} + \frac{2627\tilde{\theta}^3x}{16384\sigma t^3} + \\
&\frac{1745\tilde{\theta}^4x}{2048\sigma^3t^3} - \frac{11391\tilde{\theta}^5x}{2048\sigma^5t^3} - \frac{421\sigma^5\tilde{\theta}x}{23592960t^2} + \frac{45\sigma^3\tilde{\theta}^2x}{131072t^2} - \frac{189\sigma\tilde{\theta}^3x}{65536t^2} + \\
&\frac{695\tilde{\theta}^4x}{6144\sigma t^2} - \frac{17\tilde{\theta}^5x}{640\sigma^3t^2} - \frac{1141\tilde{\theta}^6x}{512\sigma^5t^2} + \frac{2753\tilde{\theta}^7x}{768\sigma^7t^2} + \frac{787\sigma^7\tilde{\theta}x}{251658240t} - \\
&\frac{35\sigma^5\tilde{\theta}^2x}{524288t} + \frac{1681\sigma^3\tilde{\theta}^3x}{2359296t} - \frac{1841\sigma\tilde{\theta}^4x}{294912t} + \frac{8819\tilde{\theta}^5x}{294912\sigma t} + \frac{917\tilde{\theta}^6x}{30720\sigma^3t} - \\
&\frac{6293\tilde{\theta}^7x}{15360\sigma^5t} + \frac{105\tilde{\theta}^8x}{128\sigma^7t} - \frac{385\tilde{\theta}^9x}{768\sigma^9t} + \frac{2919x^2}{262144\sigma t^6} + \frac{209\sigma x^2}{262144t^5} - \\
&\frac{105\tilde{\theta}x^2}{16384\sigma t^5} - \frac{2661\tilde{\theta}^2x^2}{1024\sigma^3t^5} - \frac{1657\sigma^3x^2}{18874368t^4} + \frac{203\sigma\tilde{\theta}x^2}{393216t^4} - \frac{51845\tilde{\theta}^2x^2}{196608\sigma t^4} - \\
&\frac{3479\tilde{\theta}^3x^2}{8192\sigma^3t^4} + \frac{116917\tilde{\theta}^4x^2}{8192\sigma^5t^4} + \frac{2171\sigma^5x^2}{188743680t^3} - \frac{251\sigma^3\tilde{\theta}x^2}{1966080t^3} - \\
&\frac{3827\sigma\tilde{\theta}^2x^2}{1966080t^3} - \frac{1955\tilde{\theta}^3x^2}{16384\sigma t^3} + \frac{50665\tilde{\theta}^4x^2}{49152\sigma^3t^3} + \frac{3585\tilde{\theta}^5x^2}{1024\sigma^5t^3} - \frac{10783\tilde{\theta}^6x^2}{768\sigma^7t^3} - \\
&\frac{1147\sigma^7x^2}{1006632960t^2} + \frac{113\sigma^5\tilde{\theta}x^2}{6291456t^2} - \frac{355\sigma^3\tilde{\theta}^2x^2}{3145728t^2} - \frac{245\sigma\tilde{\theta}^3x^2}{393216t^2} \cdots \\
&-\frac{3811\tilde{\theta}^4x^2}{131072\sigma t^2} + \frac{6935\tilde{\theta}^5x^2}{24576\sigma^3t^2} + \frac{1837\tilde{\theta}^6x^2}{61440\sigma^5t^2} - \frac{3835\tilde{\theta}^7x^2}{1536\sigma^7t^2} + \\
&\frac{9401\tilde{\theta}^8x^2}{3072\sigma^9t^2} + \frac{14121\tilde{\theta}x^3}{8192\sigma^3t^6} + \frac{11255\tilde{\theta}x^3}{65536\sigma t^5} - \frac{2283\tilde{\theta}^2x^3}{8192\sigma^3t^5} - \frac{78479\tilde{\theta}^3x^3}{4096\sigma^5t^5} + \\
&\frac{1651\sigma\tilde{\theta}x^3}{983040t^4} + \frac{1225\tilde{\theta}^2x^3}{49152\sigma t^4} - \frac{119999\tilde{\theta}^3x^3}{73728\sigma^3t^4} - \frac{601\tilde{\theta}^4x^3}{384\sigma^5t^4} + \frac{229901\tilde{\theta}^5x^3}{7680\sigma^7t^4} - \\
&\frac{11\sigma^3\tilde{\theta}x^3}{1048576t^3} + \frac{61\sigma\tilde{\theta}^2x^3}{131072t^3} - \frac{10111\tilde{\theta}^3x^3}{589824\sigma t^3} - \frac{3419\tilde{\theta}^4x^3}{12288\sigma^3t^3} + \frac{76489\tilde{\theta}^5x^3}{61440\sigma^5t^3} + \\
&\frac{8361\tilde{\theta}^6x^3}{2560\sigma^7t^3} - \frac{37267\tilde{\theta}^7x^3}{3840\sigma^9t^3} - \frac{56033x^4}{131072\sigma^3t^7} - \frac{22231x^4}{524288\sigma t^6} + \\
&\frac{13281\tilde{\theta}x^4}{65536\sigma^3t^6} + \frac{465637\tilde{\theta}^2x^4}{32768\sigma^5t^6} - \frac{4617\sigma x^4}{10485760t^5} + \frac{1163\tilde{\theta}x^4}{196608\sigma t^5} + \\
&\frac{57457\tilde{\theta}^2x^4}{49152\sigma^3t^5} - \frac{9345\tilde{\theta}^3x^4}{8192\sigma^5t^5} - \frac{944569\tilde{\theta}^4x^4}{24576\sigma^7t^5} + \frac{153\sigma^3x^4}{41943040t^4} - \\
&\frac{223\sigma\tilde{\theta}x^4}{5242880t^4} + \frac{135931\tilde{\theta}^2x^4}{7864320\sigma t^4} + \frac{6353\tilde{\theta}^3x^4}{98304\sigma^3t^4} - \frac{191603\tilde{\theta}^4x^4}{98304\sigma^5t^4} - \\
&\frac{5683\tilde{\theta}^5x^4}{4096\sigma^7t^4} + \frac{191627\tilde{\theta}^6x^4}{10240\sigma^9t^4} - \frac{180945\tilde{\theta}x^5}{32768\sigma^5t^7} - \frac{43979\tilde{\theta}x^5}{98304\sigma^3t^6} + \\
&\frac{10557\tilde{\theta}^2x^5}{8192\sigma^5t^6} + \frac{125435\tilde{\theta}^3x^5}{4096\sigma^7t^6} - \frac{49651\tilde{\theta}x^5}{7864320\sigma t^5} + \frac{2269\tilde{\theta}^2x^5}{98304\sigma^3t^5} + \\
&\frac{24451\tilde{\theta}^3x^5}{16384\sigma^5t^5} - \frac{6623\tilde{\theta}^4x^5}{6144\sigma^7t^5} - \frac{721079\tilde{\theta}^5x^5}{30720\sigma^9t^5} + \frac{1725811x^6}{1966080\sigma^5t^8} + \\
&\frac{825139x^6}{11796480\sigma^3t^7} - \frac{1003\tilde{\theta}x^6}{3072\sigma^5t^7} - \frac{364711\tilde{\theta}^2x^6}{24576\sigma^7t^7} + \frac{10303x^6}{10485760\sigma t^6} - \\
&\frac{19919\tilde{\theta}x^6}{1966080\sigma^3t^6} - \frac{227273\tilde{\theta}^2x^6}{327680\sigma^5t^6} + \frac{12363\tilde{\theta}^3x^6}{8192\sigma^7t^6} + \frac{482485\tilde{\theta}^4x^6}{24576\sigma^9t^6} + \\
&\frac{3077\tilde{\theta}x^7}{768\sigma^7t^8} + \frac{183839\tilde{\theta}x^7}{983040\sigma^5t^7} - \frac{5305\tilde{\theta}^2x^7}{8192\sigma^7t^7} - \frac{133775\tilde{\theta}^3x^7}{12288\sigma^9t^7} - \\
&\frac{363149x^8}{786432\sigma^7t^9} - \frac{330557x^8}{15728640\sigma^5t^8} + \frac{194341\tilde{\theta}x^8}{1966080\sigma^7t^8} + \frac{3781141\tilde{\theta}^2x^8}{983040\sigma^9t^8} - \\
&\frac{770581\tilde{\theta}x^9}{983040\sigma^9t^9} + \frac{91867x^{10}}{1310720\sigma^9t^{10}}
\end{aligned}
\tag{50}
$$

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