

A default copula in a lattice-based credit model

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Abstract

We introduce a default event copula into the framework of a multi-factor credit model with correlated stochastic hazard rates. We find that the codependence structure of default events in the limit of infinitesimal time steps in a finite-differencing framework is dominated by the lower tail dependence coefficient of the employed copula.

1 Introduction

It is a well known fact that mere correlation of hazard rates of underlyings, but otherwise independent default events, in a multi-dimensional Cox process setting, fails to generate sufficient codependence between default events to match market prices for first-to-default swaps, or n -th to default products in general. In order to attain market realistic levels, one needs to consider *explicit default event correlation*. In the following, we analyze how this can be done in a fully dynamic model by the aid of a generic event copula in a finite-differencing setting, and what the implications are for the net codependence of defaults over a multi-step horizon. Our focus is thus mainly on the small time step limit, as is often of importance in a finite-differencing framework, since small time steps may be necessary for reasons of numerical stability. Since we shall see that, in the limit of infinitesimal time steps, net codependence over a long time horizon may significantly differ from the codependence structure applied over a *single* time step, we are particularly interested in criteria that enable us to judge whether any one copula can, in principle, at least to some degree, preserve correlation in the multi-step limit. Whilst our focus is thus mainly on finite-differencing implementations, we emphasize that the considerations are generically applicable, both to backward and to forward induction methods, i.e., Monte Carlo simulations. The key comparison is between short time steps and single, long, time steps. First, however, we introduce the fundamental modelling assumptions, specify the employed Cox processes and how a copula function can be used to generate the full default codependence structure over a single time step.

Key words and phrases. Finite-differencing credit model, multi-name codependent default model.

^aOTC Analytics and VTB Capital

^bCommerzbank

^cSwiss Re

^dVTB Capital

2 Partial (integro) differential equations

The multi-dimensional Cox process we are going to work with throughout the following is defined by choosing displaced diffusion Black-Karasinski dynamics [Rub83, BK91] for each individual hazard rate¹

$$\lambda_i(t) = \frac{\hat{\lambda}_i(t)}{\beta_i} e^{-\frac{\beta_i^2}{2} \cdot \mathbb{V}[y_i(t)] + \beta_i \cdot y_i(t)} + \left(1 - \frac{1}{\beta_i}\right) \hat{\lambda}_i(t) , \quad (2.1)$$

with driving Ornstein-Uhlenbeck processes

$$dy_i(t) = -a_i \cdot y_i(t) dt + \sigma_i(t) dW_i(t) , \quad (2.2)$$

and correlated Brownian motions $d\langle W_i(t), W_j(t) \rangle = \rho_{i,j} dt$. The displacement parameter β_i allows us to capture the implied volatility skew on credit default swaptions, whilst $\sigma_i(\cdot)$ is typically calibrated to match the term-structure of at-the-money volatilities. In order to avoid negative hazard rates, which in turn would imply negative default probabilities, we typically set $\beta \geq 1$. We further use the time dependent parameter $\hat{\lambda}_i(\cdot)$ to calibrate the model against credit-default swaps. Whilst this framework can easily be extended to include other asset classes, we solely focus on credit specific aspects here.

Modelling credit events using a hazard rate approach essentially introduces an instantaneous jump from a non-defaulted into a defaulted state. Thus, in order to price credit dependent options using the fundamental theorem of asset pricing

$$V(t, x) = N(t) \cdot \mathbb{E}^{\mathcal{N}} [V(T) \cdot N^{-1}(T) | \mathcal{F}_t] , \quad (2.3)$$

with risk-neutral measure \mathcal{N} associated to numeraire N , we have to apply a generalised version of the theorem of Feynman-Kac [CT04, Øks03], leading to a partial (integro) differential equation,

$$\partial_t V + \mathcal{A}V + \mathbb{E}^{\mathbb{Q}} [V(t, y + \Delta y) - V(t, y) | y] - rV = 0 , \quad (2.4)$$

where we choose \mathcal{N} to be the money market measure \mathbb{Q} induced by the domestic short rate r and \mathcal{A} to denote the generator of the diffusion of the underlying Markovian drivers.

Given this basic modelling framework, we can now use a Monte Carlo simulation approach for numerical implementation. Since many first-to-default baskets and similar products involve only a small number m of underlyings ($m \lesssim 5$ or so), we can also use finite-differencing techniques for such contracts. The lattice based approach makes it comparatively easy to include early exercise features which are so commonplace in fixed income derivatives.

The standard implementation of a multi-factor stochastic hazard rate model defaultable underlyings via partial (integro) differential equations is to use an extension of the state-space to monitor the default of each underlying. Thus, not counting the stochasticity of the, say, m hazard rates, we are left with a grid consisting of 2^m states and each node $\omega \in \{0, 1\}^m$ holds information about whether the underlying is defaulted, with 0 representing default, and 1 representing non-default. To give an example, the node $\omega = (0, 1) \in \{0, 1\}^2$ indicates that underlying #1 is defaulted, and underlying #2 is not in default. Throughout the following we are going to denote the set of defaulted underlyings by I_D and the set of non defaulted ones by I_S , with $I_S \cap I_D = \emptyset$. Next, in order to discretise the integro differential part in equation (2.4), we have to specify the instantaneous default probabilities to connect the nodes during the rollback on the lattice. Typically, one chooses the jump measure according to

$$\nu(\Delta y_i, \Delta) = 1_{\{y_i=0\}} \cdot (\lambda_i \Delta \cdot \delta(\Delta y_i - 1) + (1 - \lambda_i \Delta) \cdot \delta(\Delta y_i)) , \quad (2.5)$$

¹Note that whilst (displaced) lognormal short rates are known to be a problem, (displaced) lognormal hazard rates are perfectly legitimate, since we are not computing quantities of the form $\mathbb{E} \left[e^{\int_t^T \lambda_i(u) du} \right]$.

where δ refers to the Dirac-delta function, and Δ is the time step. This effectively means that from a non-defaulted state we default with probability $\lambda_i\Delta$, whilst we stay non-defaulted with probability $1 - \lambda_i\Delta$. Thus, the discretised default probabilities, using the Fokker-Planck equivalent of equation (2.4) and ignoring the generator \mathcal{A} as well as the source term, read

$$\mathbb{P}(\omega_i(T) = 1) = (1 - \lambda_i\Delta) \cdot \mathbb{P}(\omega_i(T - \Delta) = 1) , \quad (2.6)$$

$$\mathbb{P}(\omega_i(T) = 0) = \lambda_i\Delta \cdot \mathbb{P}(\omega_i(T - \Delta) = 1) + \mathbb{P}(\omega_i(T - \Delta) = 0) , \quad (2.7)$$

where λ_i is the hazard rate on this given node for underlying $\#i$. This coupling implicitly defines the probability for multiple defaults happening over the same time step Δ to be given by the product of individual default probabilities

$$\prod_{i \in I_D(T) \cap I_S(T-\Delta)} \lambda_i\Delta . \quad (2.8)$$

Evidently, this means that, to order $\mathcal{O}(\Delta)$, this excludes joint defaults. This corresponds to the well known result that independent multi-variate Poisson processes, almost surely, have no *simultaneous* events.

Another way of viewing expression (2.8) is to say that, conditional on a specific hazard rate realization, defaults over the next discrete time step Δ , are independent. For a practical credit model, however, it would be desirable to allow for a certain degree of codependence of defaults over any one time step, especially if the time step discretisation is coarse (e.g., $\Delta \approx 1/12$) for reasons of numerical performance. For this purpose, we extend the concept using a copula to link the nodes over a given time step. Using the set of defaulted and non-defaulted underlyings I_S and I_D and introducing the following notation

$$\mathbb{P}(I_D, I_S) = \mathbb{P}(\{\omega_i = 0, i \in I_D, \omega_j = 1, j \in I_S, \forall i, j\}) \quad (2.9)$$

we link the defaults via

$$\mathbb{P}(I_D, \emptyset) = C(\mathbf{u}(I_D)) \quad (2.10)$$

where C is a Copula and the components of the vector $\mathbf{u} \in [0, 1]^m$ are given by

$$u_i = \begin{cases} \lambda_i\Delta & \text{if } i \in I_D \\ 1 & \text{else.} \end{cases} \quad (2.11)$$

We refer the reader to [Jäc02, ELM01, Nel98, CLV04] for more details about copulae and their applications in finance.

In figure 1, we illustrate the coupling induced by the use of the default copula for the three-dimensional case. Note that, if node ω has k non-defaulted states, it is connected to 2^k possible default states. Since the copula (2.10) gives us the cumulative probability of default, we have to work out the default densities via the recursive relation

$$\mathbb{P}(I_D, I_S) = \mathbb{P}(I_D \cup \{j\}, I_S) + \mathbb{P}(I_D, I_S \cup \{j\}) , \quad (2.12)$$

with $\{j\} \notin I_D \cup I_S$. We also know the event probability of all underlyings defaulting

$$\mathbb{P}(I, \emptyset) = C(\lambda_1\Delta, \dots, \lambda_m\Delta) . \quad (2.13)$$

Using the recursive relation (2.12) as well as the initial condition (2.13), one can further show that

$$\mathbb{P}(I_D, I_S) = \sum_{\substack{I_{S'} \subset I_S \\ I_{D'} = I_D \cup I_{S'}}} (-1)^{(|I_D| - |I_{D'}|)} \cdot \mathbb{P}(I_{D'}, \emptyset) \quad (2.14)$$

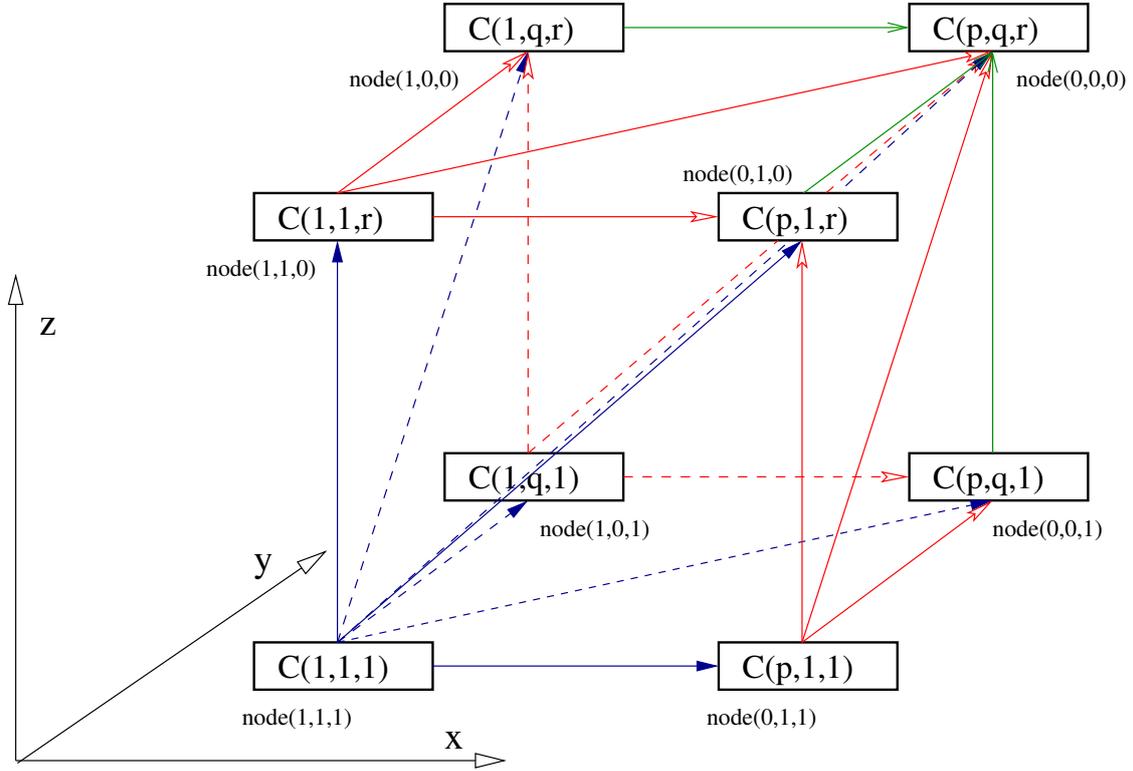


Figure 1: Default transitions. The directions on the arrows indicate the one way relation between defaulted and non-defaulted states.

where $I_D \cup I_S = I$ [GLN⁺01, theorem 2, page 7].

Another important aspect with regard to using a copula in (2.10) is the restriction that

$$\lambda_i \Delta < 1, \quad (2.15)$$

which in particular for stochastic hazard rates needs further consideration. Here, we use the approximation

$$p_i := 1 - \frac{1}{1 + \lambda_i \Delta (1 + \lambda_i \Delta)} = \lambda_i \Delta + \mathcal{O}(\Delta^3), \quad (2.16)$$

inspired by a Padé approximation of $1 - e^{-\lambda_i \Delta}$ for the default probability over the time step. It is straightforward to show that (2.16) is guaranteed to satisfy $p_i \in [0, 1]$ for all models with positive hazard rates. Since we allow greater flexibility for the dynamics of the underlying hazard rate in (2.1) using a displaced diffusion, we have to take extra care in the case of β less than 1, by adding the restriction

$$\lambda_i > 0. \quad (2.17)$$

This means we are effectively working with a displaced Black-Karasinski model with a truncating boundary at $\lambda_i = 0$ when $\beta_i < 1$. Since the Black-Karasinski model is always calibrated numerically, this poses no extra operational difficulty.

3 Computing the default transition probabilities

We have previously explained that, in principle, any m -dimensional copula can be used to generate the transition probabilities $\pi_{\omega:\omega'}$ between base state ω and transition state ω' conditional on a single realization of all m stochastic hazard rates. It is in general conceivable, and possible, that one uses a different copula for each base state. This would make it possible, for instance, to design a model that

shows *accelerated default codependence* by choosing the transition copula such that its codependence parameter, whichever way it may be defined, is an increasing function of $|I_D|$ (i.e., the number of elements of the base state vector ω that indicate default). In the following, however, we simplify matters by describing a procedure that generates all $2^m \times 2^m/2$ codependent default transition probabilities from one copula function that represents the cumulative probability that, if initially all reference entities are non-defaulted, at the end of the time step some names have defaulted with certainty, and all remaining names may or may not have defaulted. We will expand on this below.

For a practical implementation, it is helpful to think of the elements of the default state vector ω as the binary digits of an integer ω . Since the default state is absorbing, transitions between ω and ω' are not possible if any binary digits of ω are in a defaulted state that are non-defaulted in ω' . In other words,

$$\pi_{\omega:\omega'} = 0 \quad \text{if} \quad (\omega \vee \omega') > \omega \quad (3.1)$$

where $(\cdot \vee \cdot)$ represents the bitwise OR operator. Let us now define

$$c_{\omega'} = C(\mathbf{u}(\omega')) \quad (3.2)$$

with

$$u_i = \begin{cases} p_i & \text{if } (\omega' \wedge 2^i) = 0 \\ 1 & \text{else,} \end{cases} \quad (3.3)$$

using the bitwise AND operator defined as $(\cdot \wedge \cdot)$, and p_i given by (2.16). The function $c_{\omega'}$ denotes the probability that, given initially no default throughout, at the end of the time step, all those names have defaulted whose associated bit in the integer ω' is in state 0, irrespective of what may have happened to all the remaining names, i.e., they may have defaulted $[(\omega' \wedge 2^i) = 0]$ or not $[(\omega' \wedge 2^i) \neq 0]$. In other words, $c_{\omega'}$ represents the cumulative probability over both defaulted and non-defaulted states for all names $\#i$ for which $(\omega' \wedge 2^i) \neq 0$ and certainty of default for all others. With this in mind, and with the definition $\omega_0 = (1, \dots, 1) \in \{0, 1\}^m$ we can expand the recursive relationship (2.12) into the linear rule

$$\pi_{\omega_0:\omega'} = c_{\omega'} - \sum_{\omega''=0}^{\omega'' < \omega'} \pi_{\omega_0:\omega''} \cdot \mathbf{1}_{\{(\omega' \vee \omega'') \leq \omega'\}}. \quad (3.4)$$

This establishes the transition probabilities out of state ω_0 .

For the remaining transition probabilities out of all other states $\omega \neq \omega_0$, as mentioned above, we reuse the same copula information that was already computed for $\pi_{\omega_0:\omega'}$. We condition on the base state and, taking into account (3.1), compute

$$\pi_{\omega:\omega'} = \mathbf{1}_{\{(\omega \vee \omega') \leq \omega\}} \cdot \left(c_{\omega'} - \sum_{\omega''=0}^{\omega'' < \omega'} \pi_{\omega_0:\omega''} \cdot \mathbf{1}_{\{(\omega' \vee \omega'') \leq \omega'\}} \right) / c_{\omega}. \quad (3.5)$$

This simplifies to

$$\pi_{\omega:\omega'} = \mathbf{1}_{\{(\omega \vee \omega') \leq \omega\}} \cdot \pi_{\omega_0:\omega'} / c_{\omega}. \quad (3.6)$$

With this procedure, we can populate the full set of $\sim 2^m \times 2^m/2$ transition probabilities by evaluating the copula function $2^m - 1$ times (c_{ω_0} is always 1), and of the order of 2^{2m} further multiplications and additions. Since a copula function may involve exponential or power functions, avoiding its evaluation for each element of the matrix can provide a significant computational saving.

As an example, consider the case $m = 3$. In this situation, the coefficients c_{ω} are

ω	$7 = 111_2$	$6 = 110_2$	$5 = 101_2$	$4 = 100_2$	$3 = 011_2$	$2 = 010_2$	$1 = 001_2$	$0 = 000_2$
c_ω	$C(1, 1, 1) = 1$	$C(p_1, 1, 1)$	$C(1, p_2, 1)$	$C(p_1, p_2, 1)$	$C(1, 1, p_3)$	$C(p_1, 1, p_3)$	$C(1, p_2, p_3)$	$C(p_1, p_2, p_3)$

and the full matrix $\pi_{\omega:\omega'}$ is:

$\omega \backslash \omega'$	$7 = 111_2$	$6 = 110_2$	$5 = 101_2$	$4 = 100_2$	$3 = 011_2$	$2 = 010_2$	$1 = 001_2$	$0 = 000_2$
$7 = 111_2$	$1 - \pi_{7:6} - \pi_{7:5} - \pi_{7:4}$ $-\pi_{7:3} - \pi_{7:2} - \pi_{7:1} - \pi_{7:0}$	$c_6 - \pi_{7:4}$ $-\pi_{7:2} - \pi_{7:0}$	$c_5 - \pi_{7:4}$ $-\pi_{7:1} - \pi_{7:0}$	$c_4 - \pi_{7:0}$	$c_3 - \pi_{7:2}$ $-\pi_{7:1} - \pi_{7:0}$	$c_2 - \pi_{7:0}$	$c_1 - \pi_{7:0}$	c_0
$6 = 110_2$	0	$\pi_{7:6}/c_6$	0	$\pi_{7:4}/c_6$	0	$\pi_{7:2}/c_6$	0	$\pi_{7:0}/c_6$
$5 = 101_2$	0	0	$\pi_{7:5}/c_5$	$\pi_{7:4}/c_5$	0	0	$\pi_{7:1}/c_5$	$\pi_{7:0}/c_5$
$4 = 100_2$	0	0	0	$\pi_{7:4}/c_4$	0	0	0	$\pi_{7:0}/c_4$
$3 = 011_2$	0	0	0	0	$\pi_{7:3}/c_3$	$\pi_{7:2}/c_3$	$\pi_{7:1}/c_3$	$\pi_{7:0}/c_3$
$2 = 010_2$	0	0	0	0	0	$\pi_{7:2}/c_2$	0	$\pi_{7:0}/c_2$
$1 = 001_2$	0	0	0	0	0	0	$\pi_{7:1}/c_1$	$\pi_{7:0}/c_1$
$0 = 000_2$	0	0	0	0	0	0	0	1

4 Short time step limit comparison

A practical aspect of the use of a default copula in a finite-differencing application is whether the codependence behaviour of joint default, or survival, changes *structurally* when the number of steps that amount to a total time horizon, say T , is taken to the limit, assuming constant hazard rates. Since the default state is absorbing, we use as a criterion for the codependence structure the joint probability of survival over the T -horizon,

$$\mathbb{P}(\omega_1(T) = 1, \dots, \omega_n(T) = 1) = \mathbb{P}(\tau_1 > T, \dots, \tau_n > T), \quad (4.1)$$

where τ_i is the default time of underlying $\#i$. Considering the two-dimensional case, and using equation (2.14), we have

$$\mathbb{P}(\omega_1(T) = 1, \omega_2(T) = 1) = \mathbb{P}_\Delta(\emptyset, \{1, 2\}) \cdot \mathbb{P}(\omega_1(T - \Delta) = 1, \omega_2(T - \Delta) = 1) \quad (4.2)$$

$$= \mathbb{P}_\Delta(\emptyset, \{1, 2\})^n \cdot \mathbb{P}(\omega_1(0) = 1, \omega_2(0) = 1), \quad (4.3)$$

where

$$\mathbb{P}_\Delta(\emptyset, \{1, 2\}) = 1 - C(\lambda_1 \Delta, 1) - C(1, \lambda_2 \Delta) + C(\lambda_1 \Delta, \lambda_2 \Delta) \quad (4.4)$$

$$= 1 - \lambda_1 \Delta - \lambda_2 \Delta + C(\lambda_1 \Delta, \lambda_2 \Delta) \quad (4.5)$$

and $\Delta = T/n$ for n steps over a horizon from $t = 0$ to $t = T$. Note that in this section we set the short-term default probabilities $p_i \approx \lambda_i \Delta$ instead of equation (2.16) since we are interested in the limit $\Delta \rightarrow 0$, and in this limit (2.16) becomes indistinguishable from $\lambda_i \Delta$.

In the following, we discuss a small set of key copulae, and analyze how the joint probability of survival is rendered for each one of them as we take the limit $n \rightarrow \infty$ which is equivalent to $\Delta \rightarrow 0$. Before we do this, we mention that, due to the fact that the *individual marginal* default, as well as survival, probabilities, i.e.,

$$\mathbb{P}(\{i\}, \emptyset) = 1 - \mathbb{P}(\emptyset, \{i\}) \quad \forall i, \quad (4.6)$$

are not affected by the choice of codependence copula, in the two-dimensional case, our understanding of the joint survival probability behaviour immediately gives us knowledge about the joint default probability behaviour because of

$$\mathbb{P}(\{1, 2\}, \emptyset) = 1 - \mathbb{P}(\emptyset, \{1\}) - \mathbb{P}(\emptyset, \{2\}) + \mathbb{P}(\emptyset, \{1, 2\}). \quad (4.7)$$

4.1 Independence Copula

The independence copula is given by

$$C_{\text{Independence}}(u, v) = u \cdot v . \quad (4.8)$$

Here, we have

$$\mathbb{P}_{\Delta}(\emptyset, \{1, 2\}) = 1 - \lambda_1 \Delta - \lambda_2 \Delta + \lambda_1 \lambda_2 \Delta^2 . \quad (4.9)$$

Using $\Delta = T/n$, we obtain for the limit

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n = \lim_{n \rightarrow \infty} \left(1 - \lambda_1 \frac{T}{n} - \lambda_2 \frac{T}{n} + \lambda_1 \lambda_2 \left(\frac{T}{n} \right)^2 \right)^n = e^{-(\lambda_1 + \lambda_2)T} , \quad (4.10)$$

which is the same as if we evaluate the same copula over a single time step with the marginal survival probabilities $e^{-\lambda_1 T}$ and $e^{-\lambda_2 T}$, i.e.,

$$\begin{aligned} \mathbb{P}(\tau_1 > T, \tau_2 > T) &= 1 - C_{\text{Independence}}(1 - e^{-\lambda_1 T}, 1) - C_{\text{Independence}}(1, 1 - e^{-\lambda_2 T}) \\ &\quad + C_{\text{Independence}}(1 - e^{-\lambda_1 T}, 1 - e^{-\lambda_2 T}) \end{aligned} \quad (4.11)$$

$$= e^{-\lambda_1 T} \cdot e^{-\lambda_2 T} . \quad (4.12)$$

We note that the independence copula readily extends to arbitrary dimensions via

$$C_{\text{Independence}}(\mathbf{u}) = \prod_{i=1}^m u_i . \quad (4.13)$$

4.2 Perfect (co-)dependence

A copula that represents *perfect codependence*, also known as *comonotonicity*, can be formulated as

$$C_{\text{Comonotonic}}(\mathbf{u}) = \min(u_1, \dots, u_m) . \quad (4.14)$$

The limit of the multi-step survival probability is

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n &= \lim_{n \rightarrow \infty} \left(1 - \lambda_1 \frac{T}{n} - \lambda_2 \frac{T}{n} + \min(\lambda_1 \frac{T}{n}, \lambda_2 \frac{T}{n}) \right)^n \\ &= \min(e^{-\lambda_1 T}, e^{-\lambda_2 T}) . \end{aligned} \quad (4.15)$$

This is the same as if we evaluate the same copula over a single time step with the marginal survival probabilities $e^{-\lambda_1 T}$ and $e^{-\lambda_2 T}$, i.e.,

$$\begin{aligned} \mathbb{P}(\tau_1 > T, \tau_2 > T) &= 1 - C_{\text{Comonotonic}}(1 - e^{-\lambda_1 T}, 1) - C_{\text{Comonotonic}}(1, 1 - e^{-\lambda_2 T}) \\ &\quad + C_{\text{Comonotonic}}(1 - e^{-\lambda_1 T}, 1 - e^{-\lambda_2 T}) \end{aligned} \quad (4.16)$$

$$= \min(e^{-\lambda_1 T}, e^{-\lambda_2 T}) . \quad (4.17)$$

4.3 Negative dependence

A copula that represents *perfect negative codependence*, also known as *countermonotonicity*, can, in two dimensions, be formulated as

$$C_{\text{Countermonotonic}}(u, v) = (u + v - 1)_+ . \quad (4.18)$$

It is worth noting that its generalization to more than two dimensions,

$$\left(\sum_i u_i + 1 - m \right)_+, \quad (4.19)$$

is *not* a copula. To prove this, one may compute the probability assigned to the uppermost sub-half-hypercube $[1/2, 1]^m$ using (2.14). In three dimensions, we obtain

$$\begin{aligned} \mathbb{P}([1/2, 1]^3) &= (1 + 1 + 1 + 1 - 3)_+ & (4.20) \\ &\quad - (1 + 1 + 1/2 + 1 - 3)_+ - (1 + 1/2 + 1 + 1 - 3)_+ - (1/2 + 1 + 1 + 1 - 3)_+ \\ &\quad + (1 + 1/2 + 1/2 + 1 - 3)_+ + (1/2 + 1/2 + 1 + 1 - 3)_+ + (1/2 + 1 + 1/2 + 1 - 3)_+ \\ &\quad - (1/2 + 1/2 + 1/2 + 1 - 3)_+ \\ &= 1 - 1/2 - 1/2 - 1/2 + 0 + 0 + 0 - 0 \\ &= -1/2 & (4.21) \end{aligned}$$

It is straightforward to show that the probability assigned to the uppermost sub-half-hypercube, if computed assuming (4.19) as the copula, turns out to be $1 - \frac{m}{2}$ and thus is negative for all $m > 2$, which invalidates (4.19) as a copula in more than two dimensions.

It is intuitively easy to see that mutually perfect negative dependence is unattainable for more than two entities. After all, if A is exactly the opposite of B, and B is exactly the opposite of C, in some sense, then, invariably, A must be the same as C, which means that A and C are not opposite. In terms of linear correlation, this relates to the well known result that the lower limit for the correlation coefficient in a homogeneous correlation matrix is $-\frac{1}{m-1}$. Whilst mutual antidependence is a tricky concept, and whilst (4.19) does not comprise a valid copula in more than two dimensions, there are, however, ways to generalize the concept of a certain degree of antidependence to higher dimensions such as suggested by Kettler in 2008 [Ket08]. The copula suggested there is, however, difficult to visualize as a density in more than two dimensions since it involves derivatives of the Dirac function and can, possibly, only be understood in a measure theoretical sense. Here, we mention another possibility for a negative dependence copula, namely

$$C_{\text{negative dependence}}(\mathbf{u}) = \left(\sum_{i=1}^m u_i^{\frac{1}{m-1}} + 1 - m \right)_+^{m-1}. \quad (4.22)$$

This particular copula concentrates all of the joint probability density

$$\frac{\partial^m}{\partial u_1 \cdots \partial u_m} C_{\text{negative dependence}}(\mathbf{u}) = \delta \left(\sum_{i=1}^m u_i^{\frac{1}{m-1}} + 1 - m \right) \cdot \frac{(m-1)!}{(m-1)^m} \cdot \prod_{i=1}^m u_i^{\frac{1}{m-1}-1} \quad (4.23)$$

on the hypersurface defined by

$$\sum_{i=1}^m u_i^{\frac{1}{m-1}} = m - 1. \quad (4.24)$$

Evidently, this means that, conditional on $m-2$ of the uniform marginal values, one of the two remaining uniform variates is a convex function of the other. In three dimensions, for instance, we have

$$\sqrt{u_2} = 2 - \sqrt{u_1} - \sqrt{u_3}. \quad (4.25)$$

We show the surface on which all the probability density is concentrated for the three-dimensional version of (4.22) in figure 2.

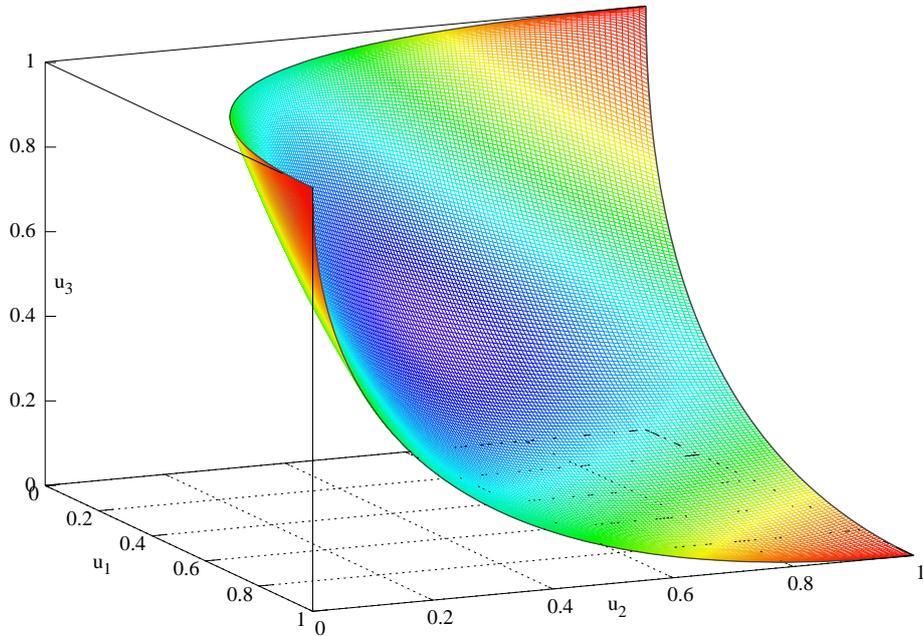


Figure 2: The Dirac density surface defined by copula (4.22) for $m = 3$. The colour temperature is an indication for the relative density on the surface, with the highest density in the centre.

Negative dependence is for most credit modelling of no particular importance since, usually, strong codependence is the more market relevant scenario. We included a discussion of negative dependence for the sake of completeness, and to show that the multi-underlying credit modelling framework presented in this article allows for this case, too, unlike conventional common factor copula models. Also, it is not entirely inconceivable that negative dependence may perhaps be desirable in a model if credit baskets such as Asda, Lidl, Aldi, and Argos (in the UK) are considered.

The limit of the multi-step survival probability for negative dependence is

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n &= \lim_{n \rightarrow \infty} \left(1 - \lambda_1 \frac{T}{n} - \lambda_2 \frac{T}{n} + (\lambda_1 \frac{T}{n} + \lambda_2 \frac{T}{n} - 1)_+\right)^n \\ &= e^{-(\lambda_1 + \lambda_2)T} \end{aligned} \quad (4.26)$$

which is the same as for the independence copula. This highlights once more how fragile negative dependence is, and serves as a warning that, if any negative dependence is to be evaluated in a financial context as a limiting case, it inevitably has to be done with a view on *long* time steps.

4.4 Copula bounds

Any copula function must lie between (4.19) and (4.14), i.e.,

$$\left(\sum_i u_i + 1 - m\right)_+ \leq C(\mathbf{u}) \leq \min(u_1, \dots, u_m). \quad (4.27)$$

This was first shown by Hoeffding [Hoe40], and independently, later, by Fréchet [Fré51]. It is for this reason that $(\sum_i u_i + 1 - m)_+$ is sometimes referred to as the *lower Fréchet-Hoeffding bound*, and the comonotonic copula (4.14) as the *upper Fréchet-Hoeffding bound*.

Since we have already computed the multi-step limit for the upper and lower bound copulae in sections 4.2 and 4.3, respectively, we now proceed to some specific parametric examples.

4.5 Ali-Mikhail-Haq Copula

The Ali-Mikhail-Haq copula [AMH78] belongs to the class of Archimedian copulae and is generated by

$$\phi_{\text{Ali-Mikhail-Haq}}(u) = \ln \left(\frac{1 - \theta(1 - u)}{u} \right). \quad (4.28)$$

The copula is given as

$$C_{\text{Ali-Mikhail-Haq}}(u, v) = \frac{u \cdot v}{1 - \theta(1 - u)(1 - v)}, \quad (4.29)$$

with $\theta \in [-1, 1]$. Interesting special cases for the Ali-Mikhail-Haq copula are the independence copula for $\theta = 0$ and the limiting case for $\theta \rightarrow 1$

$$C_{\text{Ali-Mikhail-Haq}}(u, v) = \frac{u \cdot v}{u + v - u \cdot v}. \quad (4.30)$$

If we consider the multi-step limit of the joint survival transition probabilities, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n = e^{-(\lambda_1 + \lambda_2)T}, \quad (4.31)$$

for $\theta \neq 1$ and

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n = e^{-\frac{(\lambda_1^2 + \lambda_1 \cdot \lambda_2 + \lambda_2^2)T}{\lambda_1 + \lambda_2}}, \quad (4.32)$$

for $\theta = 1$. Thus, in the limit of infinitesimal discretisation, the Ali-Mikhail-Haq copula (4.29) effectively degenerates to an independence copula (4.8).

This makes it clear that the Ali-Mikhail-Haq copula is only suitable in situations when sizeable time steps are taken. This may be the case when a significant number of underlyings are involved with a rather coarse discretisation in the spatial directions of the stochastic hazard rate drivers. In this case, the high-dimensionality of the numerical calculation means that the fastest possible evaluation of all involved analytical components is of paramount importance. Under such circumstances, and possibly only under such circumstances, the Ali-Mikhail-Haq copula may be the copula of choice.

4.6 Clayton Copula

The Clayton Copula [Cla78] is another member of the class of Archimedian copulae and its generator is

$$\phi_{\text{Clayton}}(u) = u^{-\theta} - 1. \quad (4.33)$$

The copula is given by

$$C_{\text{Clayton}}(u, v) = (u^{-\theta} + v^{-\theta} - 1)_+^{-\frac{1}{\theta}} \quad (4.34)$$

for $\theta \in [-1, \infty)$. We consider three special cases:

$$\lim_{\theta \rightarrow -1} C_{\text{Clayton}}(u, v) = C_{\text{Countermonotonic}}(u, v) \quad (4.35)$$

$$\lim_{\theta \rightarrow 0} C_{\text{Clayton}}(u, v) = C_{\text{Independence}}(u, v) \quad (4.36)$$

$$\lim_{\theta \rightarrow \infty} C_{\text{Clayton}}(u, v) = C_{\text{Comonotonic}}(u, v). \quad (4.37)$$

This wide range of possible configurations makes the Clayton copula a good workhorse in many applications.

The limit of the multi-step survival probability for the Clayton copula is

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n = e^{-\lambda_1 T - \lambda_2 T + ((\lambda_1 T)^{-\theta} + (\lambda_2 T)^{-\theta})^{-1/\theta}} = \begin{cases} e^{-(\lambda_1 + \lambda_2)T} & \text{if } \theta \leq 0 \\ e^{-(\lambda_1 + \lambda_2 - (\lambda_1^{-\theta} + \lambda_2^{-\theta})^{-1/\theta})T} & \text{else.} \end{cases} \quad (4.38)$$

This compares to the evaluation of the same copula over a single time step with the marginal survival probabilities $e^{-\lambda_1 T}$ and $e^{-\lambda_2 T}$, i.e.,

$$\mathbb{P}(\tau_1 > T, \tau_2 > T) = 1 - (1 - e^{-\lambda_1 T}) - (1 - e^{-\lambda_2 T}) - C_{\text{Clayton}}(1 - e^{-\lambda_1 T}, 1 - e^{-\lambda_2 T}) \quad (4.39)$$

$$= e^{-\lambda_1 T} + e^{-\lambda_2 T} - 1 + \left((1 - e^{-\lambda_1 T})^{-\theta} + (1 - e^{-\lambda_2 T})^{-\theta} - 1 \right)^{-\frac{1}{\theta}}. \quad (4.40)$$

We show in figure 3 how equations (4.38) and (4.40) compare for $\lambda_1 = 3\%$, $\lambda_2 = 5\%$, and $T = 5$. As

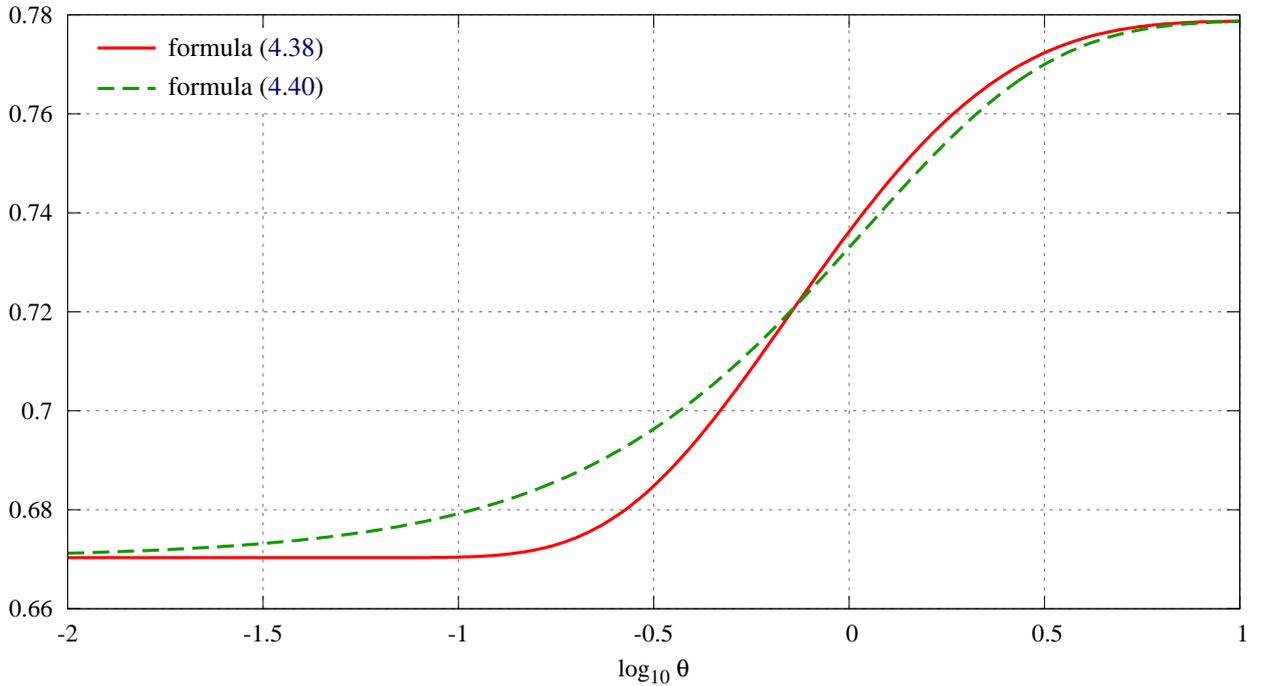


Figure 3: The multi-step limit (4.38) and the single step result (4.40) of the joint survival probability to T for the Clayton copula with $\lambda_1 = 3\%$, $\lambda_2 = 5\%$, and $T = 5$.

we can see, the curves are sufficiently similar that the multi-step use of a Clayton default copula could be adjusted, by means of calibration of θ , to perform like a single step Clayton copula.

4.6.1 Gaussian correlation coefficient mapping

Since the Clayton copula attains the same co- and counter-monotonic limits as the Gaussian copula, as well as independence, it may be helpful in practical applications to have a homotopy transformation from a Gaussian correlation coefficient ρ to the Clayton parameter θ . Intuitively, one may wish to use a transformation that establishes equivalence in a certain sense of a codependence measure. For this, we could equate a rank correlation measure for the Gaussian copula on one side, and the Clayton copula on the other. As for Spearman's rho, which is known for the Gaussian copula to be $(6/\pi) \cdot \arcsin(\rho/2)$, it turns out that its calculation for the Clayton copula involves an integral over the hypergeometric function ${}_2F_1$, which doesn't make for convenient evaluation. Kendall's tau, however, is

$$\tau_{\text{Gaussian}}^{\text{Kendall}} = \frac{2}{\pi} \arcsin \rho \quad (4.41)$$

for the Gaussian copula with correlation coefficient ρ , and

$$\tau_{\text{Clayton}}^{\text{Kendall}} = \frac{\theta}{\theta + 2} \quad (4.42)$$

for the Clayton copula. It therefore seems natural to equate the respective expressions for Kendall's τ , which is a measure for *concordance*, and thus to use the *Clayton-Gaussian concordance correspondence mapping*

$$\theta_{\text{Clayton-Gaussian Concordance Correspondence}} \hat{=} \frac{2}{\frac{\pi}{2 \arcsin \rho} - 1} \quad (4.43)$$

when it is desirable to keep the Clayton copula's co-dependence strength measured on a quasi-Gaussian scale, i.e., parametrised by an approximately equivalent (in some sense) Gaussian correlation coefficient ρ . We show an example for this in figure 4. Note that, for the chosen parameters, the continuous

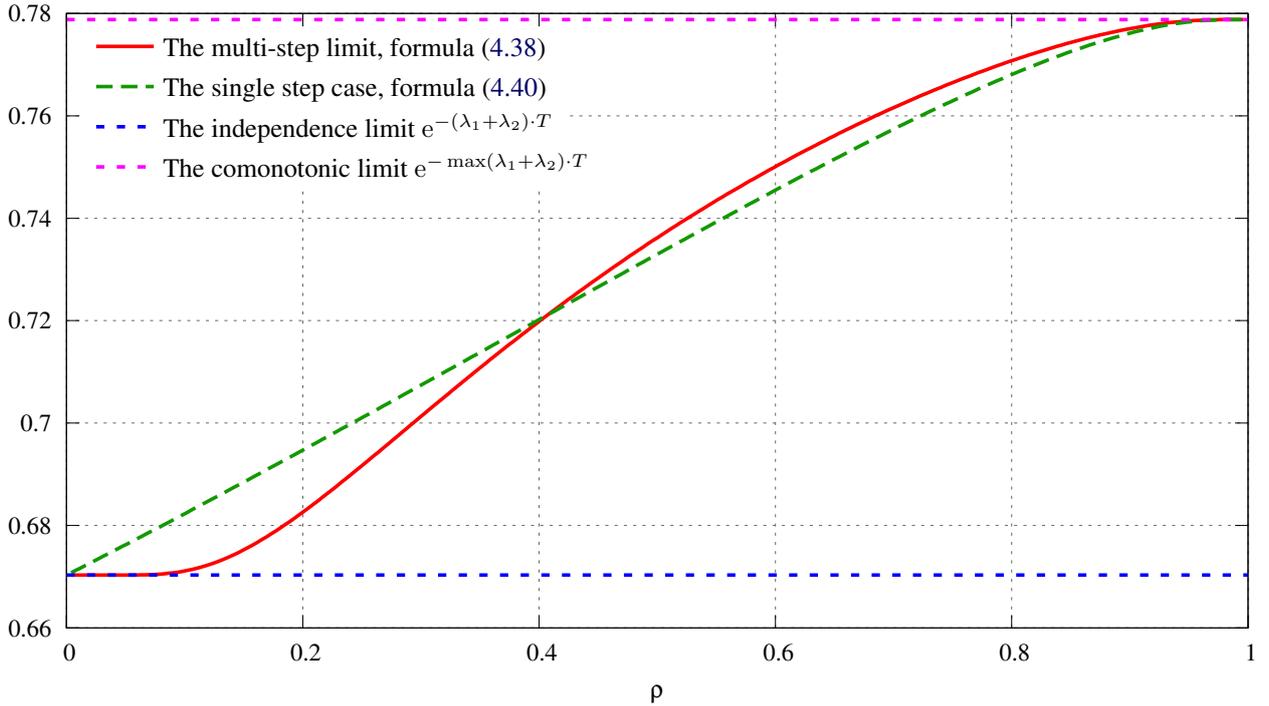


Figure 4: The joint survival probability to T for the same data as in figure 3 via the (abscissa) transformation (4.43).

time-step limit of the joint survival probability (the red line) only starts to rise from the independence limit case as of $\rho \gtrsim 0.1$.

4.6.2 The multi-dimensional case

In more than one dimension [Kim74], like for any other Archimedean copula, we have for $\mathbf{u} \in \mathbb{R}^d$

$$C(\mathbf{u}) = \phi^{-1} \left(\sum_{i=1}^d \phi(u_i) \right), \quad (4.44)$$

i.e.,

$$C_{\text{Clayton}}(\mathbf{u}) = \left(\sum_{i=1}^d u_i^{-\theta} - d + 1 \right)^{-\frac{1}{\theta}}. \quad (4.45)$$

It is important to note that, when $d > 2$, the Clayton parameter θ is limited [MN09] to

$$\theta \geq -\frac{1}{d-1} \quad (4.46)$$

which, incidentally, is the same as the lower bound for the Gaussian copula correlation coefficient:

$$\rho \geq -\frac{1}{d-1}. \quad (4.47)$$

This unfortunately implies that the lowest value for ρ that can be permitted in (4.43) for the resulting value for θ to still meet condition (4.46), is given by

$$\rho_{\text{Minimum Clayton-Gaussian Concordance Correspondence}} = \sin\left(\frac{\pi}{2(3-2d)}\right). \quad (4.48)$$

Formula (4.48) is equal to (4.47) for $d = 2$ and $d = 3$, but slightly smaller in absolute value for larger d , as is shown in figure 5. We emphasize that, whilst the restriction in ρ must be honoured if we want to

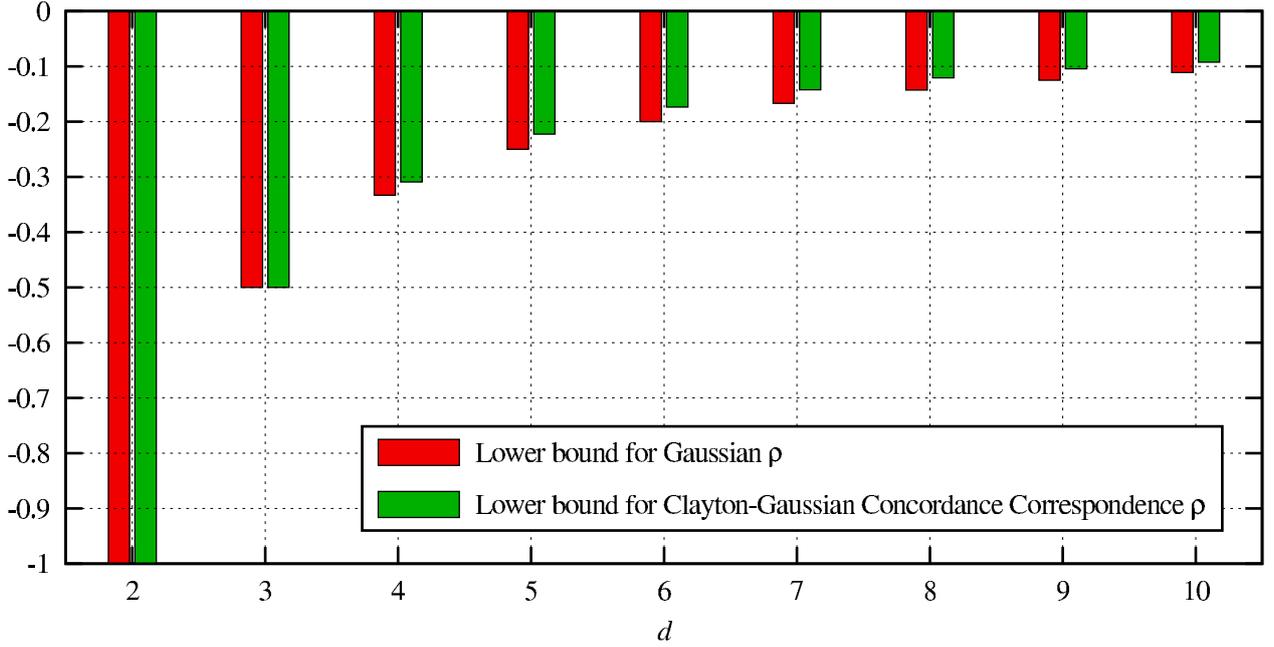


Figure 5: Lower bound for the Gaussian copula coefficient ρ as in (4.47) and the Minimum permissible Clayton-Gaussian Concordance Correspondence ρ given by equation (4.48).

use the Clayton-Gaussian concordance correspondence parametrisation (4.43), the negative correlation case is in practice rarely of significant importance, and if so, then only for benchmarking purposes, for which the restriction is of no great consequence.

4.6.3 Homotopy parametrisation

A further point of note in practical applications is that of a smooth transition from the Gaussian copula, which always comprises a reference case, to any other copula, such as the Clayton copula. For this, it is helpful to have a parametrisation that in the sense of a homotopy transformation at one end comprises the Gaussian copula, and at the other end, say, the Clayton copula. Conveniently, for this purpose, copulae have the feature that any *convex* combination of two copulae also forms a valid copula. We can therefore simply mix the Gaussian copula with the Clayton copula according to

$$C_{\text{Homotopic mix}}(\dots) = (1 - \eta) \cdot C_{\text{Gaussian}}(\dots) + \eta \cdot C_{\text{Clayton}}(\dots) \quad (4.49)$$

for $\eta \in [0, 1]$, which makes for a very convenient parametrisation with the homotopy parameter η representing the percentage of *ab-normality* of the blended copula.

4.7 Tail dependence

We have seen that the limit of the multi-step survival probability for any copula is given by

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n = \lim_{n \rightarrow \infty} \left(1 - \lambda_1 \frac{T}{n} - \lambda_2 \frac{T}{n} + C(\lambda_1 \frac{T}{n}, \lambda_2 \frac{T}{n})\right)^n.$$

If we assume for a moment, without loss of generality, that $\lambda_1 \leq \lambda_2$, we find

$$\lambda_1 \frac{T}{n} \cdot \frac{C(\lambda_1 \frac{T}{n}, \lambda_1 \frac{T}{n})}{\lambda_1 \frac{T}{n}} \leq C(\lambda_1 \frac{T}{n}, \lambda_2 \frac{T}{n}) \leq \lambda_2 \frac{T}{n} \cdot \frac{C(\lambda_2 \frac{T}{n}, \lambda_2 \frac{T}{n})}{\lambda_2 \frac{T}{n}}. \quad (4.50)$$

This enables us to derive the generic bounds

$$\boxed{e^{-(\lambda_1 + \lambda_2 - \min(\lambda_1, \lambda_2) \cdot \epsilon_L)T} \leq \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{T}{n}}(\emptyset, \{1, 2\})^n \leq e^{-(\lambda_1 + \lambda_2 - \max(\lambda_1, \lambda_2) \cdot \epsilon_L)T}} \quad (4.51)$$

with

$$\epsilon_L = \lim_{u \rightarrow 0} \frac{C(u, u)}{u} \quad (4.52)$$

being known as the *lower tail dependence coefficient of the copula* [CLV04]. Inequality (4.51) makes it clear that only copulae with non-vanishing lower tail dependence will preserve any codependence between default events in the multi-step limit $n \rightarrow \infty$.

As it is known, for instance, that the Gaussian copula for $\rho \neq 1$ has no lower tail dependence, i.e., $\epsilon_L = 0$, we can immediately state that the multi-step limit of the joint survival probability for the Gaussian copula is equal to that of the independence copula. This underlines that it is in general probably not a good idea to deploy a Gaussian copula for default codependence in a multi-step finite-differencing framework, unless it is intended to take sizeable time steps, possibly for reasons of numerical performance. In that case, however, one may be better served to use a copula that gives the additional benefit of being fast to evaluate (which the multi-dimensional Gaussian copula is not) such as the Ali-Mikhail-Haq copula mentioned in section 4.5.

For completeness, we mention that for any Archimedean copula with generator $\phi(u)$, and copula function $C(u, v) = \phi^{-1}(\phi(u) + \phi(v))$, the lower tail dependence coefficient is

$$\epsilon_L^{\text{Archimedean}} = \lim_{u \rightarrow 0} \frac{\phi^{-1}(2\phi(u))}{u} = \lim_{f \rightarrow \infty} \frac{\phi^{-1}(2f)}{\phi^{-1}(f)}. \quad (4.53)$$

For the Clayton copula, for example, this gives

$$\epsilon_L^{\text{Clayton}} = \begin{cases} 0 & \text{if } \theta \leq 0 \\ 2^{-1/\theta} & \text{else,} \end{cases} \quad (4.54)$$

which is consistent with (4.38) and (4.51).

For the Student- t copula [ELM01], in comparison, with ν degrees of freedom and homogeneous correlation matrix where all off-diagonal elements are ρ , we have

$$\epsilon_L^{C_{\nu, \rho}^t} = 2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}) \quad (4.55)$$

with $t_{\nu}(\cdot)$ being the univariate Student- t distribution function with ν degrees of freedom.

5 Conclusion

In this article we have explained how a generic copula can be used to control the codependence structure of default events in a multi-variate Cox process model for default. We presented a method to compute efficiently the discrete transition probabilities required in a numerical finite-differencing implementation. We discussed a number of candidate copulae that can be used in the given context, specifically including negative dependence, perfect codependence, and two particular Archimedean copulae, namely the Ali-Mikhail-Haq and the Clayton copula, and compared their respective merits and disadvantages. We further analyzed the limiting behaviour of the joint survival, and thus default, probabilities in a multi-step finite differencing implementation. Finally, we established lower and upper bounds for the multi-step limit of the joint survival, and thus default, probabilities for two reference entities in the given setting that are determined entirely by the chosen copula's *lower tail dependence coefficient*.

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